ALGEBRAIC CONCEPTS AND SPATIAL REASONING

Algebraic Concepts and Spatial Reasoning

Academy Introduction

This Academy is designed to provide paraeducators with the skills and knowledge necessary to assist students, grades five through eight, with mathematics skills taught in the classroom. The Academy includes the specific skill building areas of real number properties; graphical representations; algebraic concepts and problem solving; data and probability; and spatial reasoning skills as they apply to intermediate and middle school learners. The course content is designed and adapted from the standards and expectations recommended by the National Council of Teachers of Mathematics.

Note to Instructor 1: This Academy is designed on the premise that the paraeducator is working mathematically at the 5-8 grade level. Activities and timing do not allow for remediation and teaching of basic math concepts. To be successful in this Academy and gain the information at a deeper level, it is **strongly recommended that paraeducators complete the *Assisting Grades K-4 with Mathematics in the Classroom* Academy prior to completing the current Academy**. The Grades K-4 Academy should be seen as a prerequisite unless the paraeducator can demonstrate prior math knowledge.

Note to Instructor 2: You will need pattern blocks for this Academy. Resources for purchase of pattern blocks and websites for additional practice are included in the References and Product Resources section.

Module A
Instructor's Guide

Mathematics Algebraic Concepts and Spatial Reasoning

Module A: Mathematical Literacy

A. Energizer
Provide an energizer that will help participants become familiar with each other and the instructor and increase the likelihood of greater class participation. If possible, introduce a sense of anticipation and excitement about working with mathematics.

****Note to Instructor:*** If you have several participants who have attended the *Assisting Grades K-4 with Mathematics* and *Number Theory and Rational Numbers* Academies, you can decrease the time spent on Module A, as some of the material is identical to prior math Academies. See below for suggested alterations.

Alterations for Module A Goals:

- Goal 1: Complete as necessary based on group feedback; basic definition introduced may need to be reviewed
- Goal 2: Complete as necessary based on group feedback; spend time on journal organization as it is used throughout the Academy
- Goal 3: Complete as written
- Goal 4: May be skipped

B. Focus Activity: Math Journal Responses
Paraeducators will participate in an activity that will help them analyze their personal views of mathematics and their perceived knowledge and abilities. This **ungraded** introduction activity is completed prior to the course introduction because it will be biased once participants feel that the correct answers lie in the course goals being presented.

****Note to Instructor:*** This is an appropriate time to review the need for well-organized notebooks. Each participant should bring a 3-ring binder for keeping handouts, personal notes and materials used in the class. Remind and reiterate to class members that they will be taking an **assessment at the end of the class that will be an "open-book test."** The more highly organized and detailed their personal notebooks are, the more comfortable they will be with the assessment and the more likely they will be to do well. It is recommended that you bring a 3-hole punch to class for participants' use or run all handouts on 3-hole paper.

B.1 Steps

- Organize necessary materials.
- Emphasize that this activity will not be graded and will not be viewed or reviewed by other members of the group.
- Use the handout **First Thoughts About Math (H1),** providing a copy for each member of the class. Discuss with paraeducators that if they are using a 3-ring binder to organize their materials, they need to include labeled sections in which to keep specific entries.
- Ask participants to respond to the questions on the **First Thoughts About Math** handout **(H1).** They can respond using writing, drawings or any other format they are comfortable with.

First Thoughts About Math

1. How would you define mathematics?
2. What makes mathematics difficult for students?
3. How would you describe/rate you mathematical ability?

- Use transparencies **Response Chart 1** and **2 (T1/2)** for the first two questions. After participants have completed the activity (attempt to keep the time spent on this to approximately 10 minutes), ask them to share their thoughts with the rest of the class. Record their responses on the transparencies.
- Do not spend a lot of time discussing or debriefing responses to these questions. They will be revisited in the first goal.
- Share with the group that their responses are typical of others', both students and adults.

C. Academy Introduction

Using transparency and handout **Academy Goals (T3, H2)**, introduce and review the contents of the Academy.

Review the Academy goals using the following lecture information:

This Academy is designed to provide paraeducators with the skills and knowledge needed to assist students, grades five through eight, with mathematics skills taught in the classroom. Grades five through eight are pivotal grades that solidify early-elementary concepts and provide a base for high school mathematics. Due to the variety of skills developed across grades five through eight, the skills are split over two Academies.

This Academy includes the specific skill building areas of real number properties; graphical representations; algebraic concepts and problem solving; data and probability; and spatial reasoning skills as they apply to intermediate and middle school learners. The course content is designed and adapted from the standards and expectations recommended by the National Council of Teachers of Mathematics.

Two main ideas flow throughout the structure of the Academy. Part of the purpose is to build and strengthen the paraeducator's mathematics skills. While paraeducators often know some or most of the mathematics presented in this Academy, it is important that they view the activities at a deeper level of understanding. Explaining concepts to a student requires that paraeducators view activities and concepts through the eyes of a student. Therefore, they should reflect on personal experiences of working with students to explore holes in their understanding and possibly find solutions to address weak areas. It is important to look for patterns and connections throughout the Academy in order to provide comprehensive assistance to students.

The second continuous idea is the importance of fully comprehending the activities. The activities were chosen as samples of activities that paraeducators could do with students. Many of the activities contain titles that may feel quite elementary for this course. However, studies show that students retain information better from activities that can be referred to by a name such as "Move It" or "Give It a Chance" rather than just completing a worksheet. As discussions occur in the course, reference the material by referring back to the related activity. This helps make connections between concepts.

Algebraic Concepts and Spatial Reasoning in the Classroom

Module A: Mathematical Literacy
The paraeducator will:
1. Identify common misconceptions about mathematics
2. Identify the role of communication in mathematical literacy development
3. Identify the goal of problem solving and its development in the classroom
4. Compare and contrast mathematical literacy and language/reading/writing (literacy) development

Module B: Patterns and Predictions
The paraeducator will:
1. Employ strategies of problem solving to make predictions and determine the probability of an event
2. Develop integer concepts from concrete experiences
3. Develop rules for integer addition and subtraction from concrete experiences
4. Develop rules for integer multiplication and division from analyzing patterns
5. Explore the coordinate graph system

Module C: Algebraic Fundamentals
The paraeducator will:
1. Use patterns and sequences to predict and generalize outcomes
2. Describe patterns and other relationships using words and expressions
3. Relate basic patterns to algebraic concept development
4. Develop a plan for solving basic algebraic equations

Module D: Graphic Representations
The paraeducator will:
1. Explore linear and nonlinear functions as they represent data patterns
2. Interpret linear graphs as rates of change (i.e., slope)
3. Sketch and interpret graphs that represent real-life situations

Module E: Spatial Reasoning
The paraeducator will:
1. Use concrete methods to determine the connections between perimeter and area
2. Develop perimeter and area formulas for basic geometric shapes
3. Explore perimeter and area concepts in relation to circles
4. Use coordinate geometry to explain basic transformations

Goal 1: Identify common misconceptions about mathematics.

1.1 Discussion: Common Misconceptions About Mathematics

The paraeducator will define misconceptions about "mathematics" that encountered while working with students and their parents.

1.1.1 Steps

- Direct participants to return to the handout **First Thoughts (H1)** used in the focus activity while you return to the transparencies **Response Chart 1 (T1)** and **Response Chart 2 (T2)**. Briefly review the recorded answers/responses. Ask participants to review their private responses. Compare their responses to those listed in the next step.
- Use the transparency **Question 1: Common Responses (T4)**. Discuss the listed common answers and misconceptions about mathematics from both children and adults. Many other answers are possible; these are only a few that will help spark discussion:

  Question 1: Common Responses
  - ▲ Word problems (problem solving)
  - ▲ Numbers
  - ▲ Rules
  - ▲ Problems
  - ▲ Memorization
  - ▲ Skills
  - ▲ Drill
  - ▲ Homework
  - ▲ Not fun
  - ▲ Too hard

- Use the transparency **Question 2: Common Responses (T5).** Discuss the listed common answers. This is a particularly important question for those working with middle and secondary grade students as this is the time they often struggle most. Many other answers are possible; these are only a few that will help spark discussion:

  Question 2: Common Responses
  - ▲ Didn't get the "basics"
  - ▲ Vocabulary
  - ▲ Too many processes
  - ▲ Word problems
  - ▲ Not interested

- Use the transparency **Response Chart 3 (T6).** Brainstorm with attendees regarding why their personal perceptions might affect their students. Record their answers. Possible responses to look for include:
  - ▲ Only certain people can do math
  - ▲ Not successful
  - ▲ Too much vocabulary (language problems)
  - ▲ Don't see math in their everyday life like other literacy skills
  - ▲ Socio-economic-cultural background
  - ▲ Gender differences (**Note:** *Current studies show that girls are turned off to mathematics by fourth grade*)

1.2 Lecture: Defining Mathematics

Return to the transparency **Question 1: Common Responses (T4).** Ask attendees which responses on the list represent *arithmetic*. Check off responses that represent *arithmetic* (all but the last two do). Discuss with participants that common misconceptions are often the result of confusing definitions. Use the transparency **Arithmetic (T7).**

Arithmetic – Calculations Involving Predefined Rules

Arithmetic is an important part of mathematics but does not encompass all of what is included in mathematics. *Arithmetic skills* such as adding, subtracting, using fractions, algebra, etc., are markedly the reason why students/adults say they dislike *mathematics*.

Note to Instructor: Remind participants that many of the students they work with may have low *arithmetic* skills, but not necessarily low *mathematics* skills.

The above statement might confuse some participants. Another way to look at this or to examine this statement more closely would be to ask for a show of hands for the following:

"How many of you enjoy the following?"

- Doing puzzles
- Art/drawing
- Landscaping
- Cooking
- Decorating

Explain that all of these activities are mathematics activities. They all involve skills that mathematicians use. In the following modules we will be examining this much more closely. Use the transparency/handout **Mathematics Is … (T8/H3).** Discuss how Reys, Suydam, and Lindquist (1992) define mathematics. As the points of the transparency are reviewed, explain where else they will be covered in the Academy.

Mathematics Is …

Mathematics is …
1. a study of patterns and relationships (all modules)
2. a way of thinking (all modules)
3. an art (involves creativity – not just rules)
4. a language (we will spend more time on this one)
5. a tool (used in almost everything we do)

Reys, Suydam, and Lindquist (1992)

Remind participants that in order to understand mathematics as being more than arithmetic, they must recognize that math exists in almost every aspect of their lives starting when they get out of bed each morning (telling time on a clock) to choosing their clothes for the next day (probability).

Goal 2: Identify the role of communication in mathematical literacy development.

2.1 Lecture: Math as a Language

Part of understanding the definition of mathematics includes seeing math as a language.

Students have oral language before they can read. Those skills are developed from early in life. Their language skills help them to develop further literacy skills, which include word recognition, reading and comprehension. They also grow up seeing and hearing those around them using communication, reading, and writing skills. They are *obviously and observably* learning language and how to communicate, further extending the skills of reading and writing.

Learning mathematics is not an obvious process like language. Unlike language, young students do not have observable early experiences with mathematics. They, and the adults raising them, often do not recognize mathematics in use around them. Students in middle grades have even more difficulty recognizing the value and existence of the abstract mathematics they begin to learn. For these reasons, when mathematics, commonly arithmetic, is introduced, it often appears and is practiced as something outside of their regular experiences. Further complicating matters, for students for whom English is not their first language, mathematics appears to be another foreign concept to be learned.

Mathematics functions along similar lines as language. It has its own rules, terms, and symbols that require the same practice as learning a language. *Treating math as a language changes the methods by which it is taught.*

Using the **Learning a Second Language** transparency **(T9),** ask attendees what they believe is required to learn a second language. Record their responses on the transparency. Possible responses include:

- Exposure
- Practice/use
- Taking classes
- Reading and writing it
- Immersion in the culture representing the language

Now make the connection between learning a foreign language and learning mathematics. There are many similarities. Use the **Math as a Language** transparency **(T10)** to help explain math as a language.

Math as a Language

To learn the math language students/learners must:

1. Read it
2. Write it
3. Speak it
4. Do it

Read it – Implies more than just reading directions. Reading involves recognizing vocabulary, forming and interpreting math sentences and following other students' written ideas (e.g., $3 + 2 = 5$ as a sentence could be "putting together 3 cars and 2 more cars is [gives] how many?; gives me 5 cars").

Write it – Implies more than homework problems. It means explaining the problem with drawings, words, labels and vocabulary. This reinforces the reading skill.

Speak it – Teachers should not be the only ones speaking the language. Students need practice with the language. They need to hear themselves, hear models (from teachers or other adults) and hear their peers. This requires that students discuss math or provide input in mathematical discussions. This includes having children talk out their work before ever putting pencil to paper.

Do it – This requires individual practice. This means trying out the language skills in homework practice, in group conversations or in a reflective journal. Putting all of the prior skills together allows the student to use the language.

The four previous ideas include many literacy skills: reading, writing, communicating, listening, and speaking. These are not traditionally thought of as mathematics skills. If we attempt to link language skills and mathematics, we must be open to changing practices and beliefs to help students think about and use more than just the traditional methods that are typically offered. We are really talking about a change in philosophy. It is important to recognize the factors of this philosophy and why they are important.

Discuss the importance of this change of philosophy for mathematics.

- Students can link mathematics with successful prior learning.
- Students can build on language skills they may already possess and add to their language base with the language of mathematics.
- Students can better communicate their understandings with both the teacher and their peers.
- Students can begin to see the mathematics process as a work in progress when they hear and participate in the process.

This change in philosophy emphasizes the role of communication in the development of mathematical literacy. Language is an important element in literacy. The purpose of language is to develop clear communication of ideas and processes. This is as important in mathematics as it is in everyday social communication.

2.2 Activity: Creating a Math Journal

The paraeducator will continue creating his/her math journal as a model for use with students. This portion of the journal will focus and build on the need for math communication.

2.2.1 Steps

- Using the transparency and handout **Math Journal (T11/H5)**, introduce the Math Journal.
- The handouts may be used as the dividers for each section of the journal as reminders.
- Explain that this journal will consist of three sections. Participants are expected to provide their own paper for taking notes within each section. Handouts should be filed appropriately within the following organization:
  - ▲ reflections
  - ▲ problem solving
  - ▲ reference
- Discuss the first section, reflections, including the following information regarding the paraeducators' use of the journaling process.
  - ▲ Reflections often provide new insight into student understanding, often more so than homework and problem performance
  - ▲ Journals help students practice communication for mathematics
  - ▲ The **First Thoughts (H1)** handout should be filed under this heading
- The second section is for recording in-class problem-solving work.
  - ▲ This keeps participants' work together in one place and makes it easy to go back and look at strategies and processes. When paraeducators begin carrying over some of the strategies and skills they are learning with their students, they will frequently refer to this section. Being organized provides easy access and increases the likelihood that the paraeducators will make good use of learning.

- The last section of the journal is used as a reference tool.
  - ▲ Each working period, ask participants to put unfamiliar or new concepts or vocabulary in their reference book.
  - ▲ Give help with the proper spelling of the vocabulary words but have the students define the word/concept in their own words and use pictures where necessary.
  - ▲ Clarify details to make sure paraeducators' definitions are mathematically sound.
- The journal is designed to be something the participants will use on their own and should be individually organized for best personal use.
- The journal supports paraeducators when no classroom teacher is present and helps them be successful.
- Use of a Math Journal is an important practice when working with students who are second-language learners; math may seem difficult and overwhelming because of the multiple unique terms used. The paraeducator can keep a list of new learning for each student and be better prepared to seek assistance with problem areas.
- A journal also provides the opportunity for the necessary repetition of ideas.

Goal 3: Identify the goal of problem solving and its development in the classroom.

3.1 Lecture: Define Problem Solving

Use the **Problem Solving: A Definition** transparency and handout **(T12/H6).**

NCTM (2000) defines problem solving as follows:

Problem Solving: A Definition

Problem solving means engaging in a task for which the solution method is not known in advance.

In order to find a solution, students must draw on their knowledge, and through this process, they often develop new mathematical understandings.

Solving problems is not only a goal of learning mathematics, but also a major *means* of doing so.

Students should have frequent opportunities to formulate, grapple with and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking. (http://standards.nctm.org/document/chapter3/prob.htm)

While this is a lengthy definition, it includes very important points about problem solving.

- Problem solving should not be a separate unit of learning.
- Problem solving should not be confused with word problems. Many word problems are merely a rephrasing of an arithmetic problem.
- Good application problems allow for problem solving beyond the use of a simple algorithm (rule).
- Problem solving is the adhesive that holds the math curriculum together.

Young children are naturally curious and are typically very flexible when they encounter new learning. Most often they do not fear new challenges and like the excitement of solving a problem or challenge. They enjoy experimenting with new methods of problem solving and do not fear mistakes or failure. The need to solve a proposed problem generates the need for acquisition of skills. *Problems must be posed where solutions are not immediately obvious*. This encourages children to take a risk and strive to find a solution.

In contrast, middle grade students present a challenge as they often bring more negativity towards word problems due to prior poor experiences with problem-solving activities. As a result, they are less likely to experiment with solving strategies and more likely to stop the process when a basic failure arises in attempting a solution.

3.2 Activity: Problem-Solving Practice

The paraeducator will participate in an activity requiring practice of personal problem-solving skills and examination of the problem-solving skills of others.

Materials:

- Transparency **Handshake Problem (T13)**

3.2.1 Steps

- Divide the class into small groups of 4-6 people.
- Using the **Handshake Problem (T13)** transparency, pose the following question to the group:

  Handshake Problem:
  "At a party attended by 10 people, each of whom greeted all of the others by shaking hands, how many handshakes occurred?"

- Direct the groups to work through the problem. Ask someone in each group to take notes to be shared with the class. Their notes should include how the group came to answers or conclusions; what was their thinking?
- Share group answers with the entire class.

3.3 Lecture: Problem Solving and the Handshake Problem

The Hand-Shake Problem is a common counting problem in grades 5-8. It forces students to read into the problem to decide on the parameters for solving. As a group, discuss some parameters that they used:

- No one could shake hands with the same person twice
- Everyone had to shake hands at least once
- No one could shake hands with themselves

Solution: There are 45 handshakes.

The first person shakes nine times (one with each other person). The next person shakes only eight times because he cannot shake hands with himself or repeat with person Number One. The pattern decreases by one each time a person shakes more hands.

Ask participants to share their methods for solving the problem. Participants will be surprised at the variety of methods used. Some participants will draw small people with arrows showing handshakes whereas others will make a table. It is important that each participant realizes that there are multiple ways to reach a solution; there is no one correct way.

As a group, discuss which math skills/concepts are necessary to solve the problem. After soliciting responses from the group, use the transparency **Skills and Concepts (T14)** to review or to provide information. Add any other skills to the transparency.

Skills and Concepts (suggestions)

- Pattern development
- Organizing data
- Counting
- Reasoning

It is important for paraeducators to understand that as students tackle new problems, it is helpful for them to have a plan. This is an area where paraeducator can fulfill an important role in helping students. George Polya (1957) produced a quick guide to help students become good problem solvers. He offers the following suggestions (use the **Problem Solving with a Plan** transparency and handout **[T15/H7]**).

Problem Solving with a Plan

Steps for Problem Solving

1. Understand the problem
2. Devise a plan
3. Carry out the plan
4. Look back

Understand the problem

- Rephrase the problem in your own words
- What are you trying to find or do?
- Unknowns?

Devise a plan

In any math course, students need tools. Tools are not just rules. Tools involve strategies. While names are presented here, the names are not standard nor should students focus on memorizing the titles. They are simply ways to refer to processes that can be useful when solving problems. This is only a partial list for discussion:

Problem-Solving Strategies

- Look for a pattern
- Create a simpler problem
- Make a table
- Draw a diagram
- Write an equation
- Guess and check
- Work backward
- Identify a subgoal for the problem (break a problem down)

As students develop a plan, they often want to get to work right away rather than finish their plan. For young children, talking about a plan and recording it is often a good place to start developing problem-solving skills.

Carry out the plan

- Implement the strategy
- Perform any necessary computations
- Keep a record of your work

Look back

- Check results
- Revise the plan
- Make connections to other problems
- Try again if necessary

The last step is often the most difficult. Many students assume that when they get to the end, it is over. Some students frustrate easily if the problem is incorrect the first time. Part of problem solving is learning to revise and try again. This supports the old adage, "If at first you don't succeed, try, try again."

This method also very closely follows Bloom's Taxonomy. Learners go through the stages of knowledge and comprehension as they come to understand the problem; they then use application skills as they devise a plan to solve the problem. Next, they carry out the plan and begin to analyze it for success. Lastly, they look back at their activity and synthesize and evaluate their learning.

Goal 4: Compare and contrast mathematical literacy and language/reading/ writing (literacy) development.

4.1 Activity: Compare and Contrast

The paraeducator will compare and contrast mathematical literacy and language/reading (literacy) development.

Materials:

- Transparency/handout **Comparing and Contrasting Math and Reading Literacy (T16/H8)**

4.1.1 Steps

- Using the transparency/handout **Comparing and Contrasting Math and Reading Literacy (T16/H8)**, direct paraeducators to work in small groups.
- Ask them to respond to the question: "What skills/expectations do you think a student has developed in order to have the language/reading/writing literacy skills needed in grades 5-8?"
- Direct each group to assign a note taker and representative to present their responses to the class during the large-group followup portion of the activity.
- After the groups have had time to meet and discuss, ask them to share their responses with the entire class. Record their responses on the transparency. Share the group ideas to be recorded on a class list for literacy. If the recorded responses do not include the items on the list below, add them.

Comparing and Contrasting Math and Reading Literacy

Reading

- ▲ Letter recognition
- ▲ Letter sounds
- ▲ Word recognition
- ▲ Word comprehension
- ▲ Sentence production (rules)
- ▲ Sentence comprehension
- ▲ Paragraph summarization
- ▲ Order of events
- ▲ Prediction

Language

- ▲ Letter sounds
- ▲ Appropriate pronunciation
- ▲ Ability to communicate ideas
- ▲ Ability to formulate questions and responses

Writing

- ▲ Letter formation
- ▲ Sentence production
- ▲ Ability to communicate ideas

Provide class members with the transparency and handout **Defining Literacy (T17/H9).**

Defining Literacy

Literacy	Mathematical Literacy
Fundamentals (letters, sounds, etc.)	Fundamentals (numbers, symbols, etc.)
Rules (punctuation, spelling, etc.)	Rules (algorithms, order of operations, etc.)
Sentence production to represent ideas	Mathematical sentences (equations)
Practical uses outside the classroom (magazines, signs, conversations, etc.)	Practical uses outside the classroom (money, time, distance, shapes, etc.)
Comprehension	Ability to explain a problem or process
Prediction	Patterns and problem solving
Communication	Communication

- Ask the small groups to reconvene and discuss how the skills they listed previously fall into the categories on the handout: fundamentals, rules, sentence production, practical uses, comprehension, prediction and communication.
- Direct the groups to discuss the same categories of mathematical literacy and compare the fundamental concepts that are also true in this area.

4.2 Lecture/Conclusion: Defining Mathematical Literacy

The term *mathematical literacy* implies that mathematics is not a static, one-sided concept. This reflects the new philosophy promoted in many current mathematics programs starting in early elementary grades.

The **Defining Literacy** handout **(H9)** contains only a few solid comparisons for the development of both types of literacy. Success in mathematics requires the same expectations as for reading, writing, and language skills.

While individual skills may not line up exactly, foundational concepts tend to run parallel with a new definition of mathematics. Paraeducators may need help in discussing and reorganizing their thinking. This concept will be reinforced and revisited throughout this Academy.'

Module A
Handouts

First Thoughts About Math

Please respond to the following journal questions. Feel free to respond using writing, drawings or any other format with which you are comfortable.

1. How would you define mathematics?

2. What makes mathematics difficult for students?

3. How would you describe/rate your own mathematical ability?

Algebraic Concepts and Spatial Reasoning in the Classroom

Module A: Mathematical Literacy

The paraeducator will:

1. Identify common misconceptions about mathematics
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3. Identify the goal of problem solving and its development in the classroom
4. Compare and contrast mathematical literacy and language/reading/writing (literacy) development

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2. Develop integer concepts from concrete experiences
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Mathematics Is …

1. a study of patterns and relationships
2. a way of thinking
3. an art
4. a language
5. a tool

-Reys, Suydam, and Lindquist (1992)

Math as a Language

To learn the math language students/learners must:

1. Read it
2. Write it
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Read it – Implies more than just reading directions. Reading involves recognizing vocabulary, forming and interpreting math sentences and following other students' written ideas (e.g., $3 + 2 = 5$ as a sentence could be "putting together 3 cars and 2 more cars is [gives] how many?; gives me 5 cars").

Write it – Implies more than homework problems. It means explaining the problem with drawings, words, labels and vocabulary. This reinforces the reading skill.

Speak it – Teachers should not be the only ones speaking the language. Students need practice with the language. They need to hear themselves, hear models (from teachers or other adults) and hear their peers. This requires that students discuss math or provide input in mathematical discussions. This includes having children talk out their work before ever putting pencil to paper.

Do it – This requires individual practice. This means trying out the language skills in homework practice, in group conversations or in a reflective journal. Putting all of the prior skills together allows the student to use the language.

The four previous ideas include many literacy skills: reading, writing, communicating, listening and speaking. These are not traditionally thought of as mathematics skills. If we attempt to link language skills and mathematics, we must be open to changing practices and beliefs to help students think about and use more than just the traditional methods that are typically offered. We are really talking about a change in philosophy. It is important to recognize the factors of this philosophy and why they are important.

Math Journal

Reflections

- My personal responses and insights into student understanding
- Thoughts and insights about my own learning
- "AH-HA" moments; the light bulb just went on for me, and I want to remember what I learned and how I learned it
- Writing about the processes I am learning will help me communicate my learning to others

Problem Solving

- It's easier to remember how I solved a problem if I can review my own process

- Keeps my work samples organized and I can find and reference them easily

- I will be able to review my own work to help me clarify my own process when I am working with students.

Reference

- Record unfamiliar concepts
- Record definitions and other language that will be helpful later
- I can keep a list of new learning for each student
- I can be better prepared to seek assistance with problem teaching areas.

Problem Solving: A Definition

- Problem solving means engaging in a task for which the solution method is not known in advance.
- In order to find a solution, students must draw on their knowledge, and through this process, they often develop new mathematical understandings.
- Solving problems is not only a goal of learning mathematics. It is also a major means of doing so.
- Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking.

Important Points About Problem Solving

- Problem solving should not be a separate unit of learning.
- Problem solving should not be confused with word problems. Many word problems are merely a rephrasing of an arithmetic problem.
- Good application problems allow for problem solving beyond the use of a simple algorithm (rule).
- Problem solving is the adhesive that holds the math curriculum together.

Problem Solving With a Plan

Steps for Problem Solving:

1. Understand the problem
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Understand the problem
- Rephrase the problem in your own words
- What are you trying to find or do?
- Unknowns?

Devise a plan
Students need tools. Tools involve strategies. The following is a list of problem-solving tools or strategies that students can use. The strategies are simply ways to refer to processes that can be useful in solving problems. This is only a partial list, and many other strategies may be used.

Problem-Solving Strategies
- Look for patterns
- Create a simpler problem
- Make tables
- Draw diagrams
- Write equations
- Guess and check
- Work backward
- Identify a subgoal for problems (break the problem down)

Carry out the plan
- Implement the strategy
- Perform any necessary computations
- Keep a record of your work

Look back
- Check results
- Revise the plan
- Make connections to other problems
- Try again if necessary

Comparing and Contrasting Math and Reading Literacy

Instructions: Please record personal responses, small-group responses and responses from whole-class discussions in the columns below.

Reading	Language	Writing

Defining Literacy

Literacy	Mathematical Literacy
Fundamentals (letters, sounds, etc.)	Fundamentals (numbers, symbols, etc.)
Rules (punctuation, spelling, etc.)	Rules (algorithms, order of operations, etc.)
Sentence production to represent ideas	Mathematical sentences (equations)
Practical uses outside the classroom (magazines, signs, conversations, etc.)	Practical uses outside the classroom (money, time, distance, shapes, etc.)
Comprehension	Ability to explain a problem or process
Prediction	Patterns and problem solving
Communication	Communication

Module A
Transparencies

Response Chart 1

Define Mathematics

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Response Chart 2

What Makes Mathematics Difficult for Students?

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Academy Goals:
Algebraic Concepts and Spatial Reasoning in the Classroom

Module A: Mathematical Literacy
The paraeducator will:
1. Identify common misconceptions about mathematics
2. Identify the role of communication in mathematical literacy development
3. Identify the goal of problem solving and its development in the classroom
4. Compare and contrast mathematical literacy and language/reading/writing (literacy) development

Module B: Patterns and Predictions
The paraeducator will:
1. Employ strategies of problem solving to make predictions and determine the probability of an event
2. Develop integer concepts from concrete experiences
3. Develop rules for integer addition and subtraction from concrete experiences
4. Develop rules for integer multiplication and division from analyzing patterns
5. Explore the coordinate graph system

Module C: Algebraic Fundamentals
The paraeducator will:
1. Use patterns and sequences to predict and generalize outcomes
2. Describe patterns and other relationships using words and expressions
3. Relate basic patterns to algebraic concept development
4. Develop a plan for solving basic algebraic equations

Module D: Graphic Representations
The paraeducator will:
1. Explore linear and nonlinear functions as they represent data patterns
2. Interpret linear graphs as rates of change (i.e., slope)
3. Sketch and interpret graphs that represent real-life situations

Module E: Spatial Reasoning
The paraeducator will:
1. Use concrete methods to determine the connections between perimeter and area
2. Develop perimeter and area formulas for basic geometric shapes
3. Explore perimeter and area concepts in relation to circles
4. Use coordinate geometry to explain basic transformations

Question 1: Common Responses

- Word problems (problem solving)
- Numbers
- Rules
- Problems
- Memorization
- Skills
- Drill
- Homework
- Not fun
- Too hard

Question 2: Common Responses

- Didn't get the "basics"
- Vocabulary
- Too many processes
- "Word problems"
- Not interested

Response Chart 3

Your Perceived Abilities

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-
-

Arithmetic: Calculations Involving Predefined Rules

Mathematics Is ...

1. a study of patterns and relationships

2. a way of thinking

3. an art

4. a language

5. a tool

- Reys, Suydam, and Lindquist (1992)

Learning a Second Language

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Math as a Language

To learn the math language students/learners must:

1. Read it
2. Write it
3. Speak it
4. Do it

Math Journal

- ▲ Reflections
- ▲ Problem Solving
- ▲ Reference

Problem Solving: A Definition

Problem solving means engaging in a task for which the solution method is not known in advance.

In order to find a solution, students must draw on their knowledge, and through this process, they often develop new mathematical understandings.

Solving problems is not only a goal of learning mathematics, it is also a major *means* of doing so.

Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking.

NCTM, 2000

Handshake Problem

At a party attended by 10 people, each of whom greeted all of the others by shaking hands, how many handshakes occurred?

Skills and Concepts

- Pattern development
- Organizing data
- Counting
- Reasoning

Problem Solving With a Plan

Steps for Problem Solving:

- Understand the problem
- Devise a plan
- Carry out the plan
- Look back

Comparing and Contrasting Reading and Math Literacy

Reading	Language	Writing

Defining Literacy

Reading Literacy	**Mathematical Literacy**
Fundamentals	Fundamentals
Rules	Rules
Sentence production to represent ideas	Mathematical sentences
Practical uses outside the classroom	Practical uses outside the classroom
Comprehension	Ability to explain a problem or process
Prediction	Patterns and problem solving
Communication	Communication

Module B
Instructor's Guide

Module B: Patterns and Predictions

A. Module Introduction

Use the transparency **Module Goals (T1)** to review the goals of the module.

Module B: Patterns and Predictions
The paraeducator will:
1. Employ strategies of problem solving to make predictions and determine the probability of an event
2. Develop integer concepts from concrete experiences
3. Develop rules for integer addition and subtraction from concrete experiences
4. Develop rules for integer multiplication and division from analyzing patterns
5. Explore the coordinate graph system

Module B focuses on patterns and predictions. One of the major functions of mathematics is its use in predicting or representing possible outcomes in a problem-solving situation. The ability to predict involves looking for patterns. Recognizing patterns starts in early elementary as students begin to look at shapes and the calendar. Older students use patterns to begin to expand concrete arithmetic rules that they learned in the early elementary grades to more general rules that work in many mathematical situations.

Goal 1: Employ strategies of problem solving to make predictions and determine the probability of an event.

1.1 Lecture: Concept Review

Note to Instructor: Remind participants to use their Math Journals to take notes during lecture periods.

In grades K-4, students have begun to develop prediction skills as they have mastered the basic concepts of likelihood and fairness. Briefly review these concepts as necessary using the transparency/handout **Probability Review (T2/H1).** When we discuss probability, we are looking at the chance of an event occurring.

Probability:
- The chance of an event occurring
- Value between 0 and 1
- Requires the use of fractions
- Determined by the number of possible outcomes

An example of *equally likely* would be heads or tails on a coin. That is, the chance is 50/50 that either one will show up.

Equally likely:
- There is an equal *probability (chance)* for an outcome.

We consider a coin “fair” as the events are equally likely. Many carnival games are not “fair” as the target to win a prize is smaller than the area that does not produce a prize.

Fair:
- Object or event has options that are equally likely to occur.

For likelihood, use the example of rain percentages: 70% chance is likely to rain.

Likelihood:
- How probable or likely it is that something will happen.
- Likelihood is also known as **probability**.

Impossible: Event will never occur.

Certain: Event will absolutely occur.

Also review the concept that numbers may be used to show how probable an event is. For example, a probability of zero fits an event that is impossible. A probability of 1 may be equated to 100%, which means the event is certain to occur. All other levels of probability exist between 0 and 1 (which may be thought of conceptually as between 0% and 100%).

1.2 Activity: Flip It

The paraeducator will use concrete objects to perform an experiment and collect data for prediction.

Materials:
- One coin per pair

1.2.1 Steps
- Ask group members what they believe the likelihood or probability is for flipping a heads? (50% or $\frac{1}{2}$); for flipping a tails? (50% or $\frac{1}{2}$).
- Ask each pair to flip the coin 20 times and record the results.

1.3 Discussion: Making Heads or Tails of Probability

The paraeducator will use data from concrete objects to define experimental and theoretical probability.

1.3.1 Steps
- Ask group members what industries might use probability or prediction and how it might be used. Add examples as they arise (possible answers below; transparency/handout **Using Probability and Prediction [T3/H2]**).
  - ▲ Predict behaviors
  - ▲ Explain past occurrences from patterns or data

- ▲ Examples:
  - o The gaming (casino) industry relies on theoretical probability to make a profit
  - o Weather forecasters
  - o Medical industry
  - o Insurance companies

Note to Instructor: It is important to emphasize the thought process involved with probability throughout this discussion as there are too many conditions to memorize independently; this is part of the problem-solving process.

- Ensure that participants learn to properly name the likelihood of the event; work through transparency/handout **Naming Probability (T4/H3)** with the following.
- The group already determined before beginning that the probability or chance of getting a heads was 50% or ½; the same was true for tails.
- Share that P(H)=1/2, which says "the probability of getting a heads (H) is ½."
- Emphasize that each part of the fractional value has a meaning (use transparency/handout **Naming Probability [T4/H3]**).
  - ▲ Determine the *sample space (S)*

    Sample Space (S)
    Set of all possible outcomes

    - o For the coin $S=\{H,T\}$, (H=heads; T=tails; {} imply a set or grouping)
    - o This becomes the denominator of the fraction to represent the probability
  - ▲ Define the *event* you are trying to predict such as getting a heads or rolling a 1 on a single die

    Event
    Any subset or subgroup of the sample space

    - o The event becomes the numerator
    - o For the coin, heads (H) is one of two possible events
  - ▲ These concepts define the *theoretical probability,* which is a mathematical computation; A is the defined event, which can be a single event or multiple events

Theoretical Probability (defined mathematically)

$$\frac{\text{Number of elements in A}}{\text{Number of elements of sample space}} = \frac{n(A)}{n(S)}$$

Note to Instructor 1: This definition appears mathematically complex but will be simplified in this goal.

- For the coin example, the *theoretical probability* is $\frac{1}{2}$ because heads is a single occurring event (numerator: 1), and there are two possible outcomes in the sample space (denominator: 2).

Note to Instructor 2: The value $\frac{1}{2}$ should be discussed as one out of two times the event would occur; it should not be solely interpreted as only two elements in a sample space. Remind participants that fractions may be reduced as in 4/8, which would mean eight elements initially in the sample space.

In the prior activity the group did an experiment by flipping coins; this is *experimental probability* (transparency/handout **Naming Probability [T4/H3]**).

Experimental Probability
Same calculation but the data come from an experiment

- Ask each pair of paraeducators to calculate their probability for both heads and tails for the coin-flipping experiment. Ask where the data for the numerator and denominator come from.
  - ▲ Numerator: number of times the heads occurred; number of times the tails occurred
  - ▲ Denominator: number of total times that the coin was flipped (20 for each pair)
- Ask group members how they know that their fractions are correct (using common denominator rules, adding the numerators should produce a fraction of 20/20=1).
- Put all data on the board for head and tails.
- Check to see if the *experimental probability* aligns with the *theoretical probability*
  - ▲ Using the class data, calculate the total probability by adding all the times when heads appeared (looking at the numerators) and dividing by how many times the coin was flipped for the group (20 times the number of pairs)
  - ▲ May need to discuss equivalent forms of $\frac{1}{2}$ or fractions that would be considered close to $\frac{1}{2}$
  - ▲ Most likely, the data will not be 50%
- Depending on the outcome of the class probability, discuss one or more of the following concepts:
  - ▲ *Why was the experimental outcome different from the theoretical outcome?*
    - o Chance
    - o Number of flips
  - ▲ *Will the data always be so close?*
    - o No, it depends on chance
    - o No, it might be different with the number of flips

- Hypothesize what will happen to the values of the *experimental* and *theoretical probability* with more flips?
  - ▲ More flips (more data) will make the *experimental* probability approach the *theoretical* probability
- Discuss why there needs to be two types of probability.
  - ▲ Often an experiment is not possible, so *theoretical* probability is all that is available
    - o Example: probability of a child dying from a complication with a new drug
  - ▲ *Theoretical* helps check our *experimental* data for accuracy
- Looking closely at the class data, discuss the following questions. Make sure to include proper notation as shorthand for the long sentences (use the transparency/handout **Making Heads or Tails of Probability [T5/H4]**).
  - ▲ What is the probability of getting a *heads or tails* on one flip? (1 or 100%; this is certain)
    - o This may be written in the form $P(H \cup T) = 1$
    - o The "$\cup$" symbol stands for union or the word "or" when describing a set as in our example
  - ▲ What is the probability of getting a star on a coin? (0 or 0%; this is impossible)
    - o This may be written as $P(*) = 0$
  - ▲ What is the probability of getting a *heads and a tails* on one flip? (0 or 0%; this is impossible)
    - o This may be written in the form $P(H \cap T) = 0$
    - o The "$\cap$" symbol stands for intersection or the word "and" when describing a set as in our example
    - o Heads and tails are considered *mutually exclusive* (use transparency/handout **Making Heads or Tails of Probability [T5/H4]**)

Mutually Exclusive
When event A occurs, event B cannot occur

- The coin-flipping example only included two event options; in real life, probability is not that simple.
- With multiple events, probabilities for mutually exclusive events may be added together to determine the final probability. Record the following on the transparency/handout **Making Heads or Tails of Probability (T5/H4)** (do not spend a lot of time here as the next activity gives ample practice):
  - ▲ Use the example of a single die, which will be used on the homework assignment
  - ▲ All of the events (number 1-6) are mutually exclusive as they cannot occur at the same time on a single roll
  - ▲ The probability of rolling a 1 is 1/6, $P(1) = 1/6$ (it is the same for all sides)
    - o Might want to review that this is about 17% (16-2/3% to be exact)
  - ▲ The probability of rolling a 1 *and* a 2 on a single roll is 0, as they are mutually exclusive, $P(1 \cap 2) = 0$

- ▲ The probability of rolling a 1 *or* a 2 on a single roll is 1/6 + 1/6 = 2/6 =1/3, $P(1 \cup 2) = 1/3$
  - o The individual probabilities are added together as the events are mutually exclusive
- Emphasize that this explanation of probability is very basic; in grades 5-8 students begin to experience what are called *multistage experiments* (all of these are single stage), which require a great deal of organization and much more terminology than this Academy allows due to time constraints.
- Point out that students who master these basics will have little problem with more complex projects.

1.4 Activity: Give It a Chance

The paraeducator will use concrete objects to develop probability concepts.

Materials:
- Pattern blocks or chips in three different colors (red, green, blue)
- Transparency/handout **Give It a Chance (T6/H5)**
- Calculators (optional)

1.4.1 Steps
- Emphasize before beginning this activity that reading and processing are key to mastering the mathematics.
- In pairs, have participants complete the handout **Give It a Chance (H5).**
  - ▲ Participants may need assistance in remembering how to find the denominator, which is the total number of elements in the sample space or the total possible tees
  - ▲ Encourage participants to use the manipulatives to see each problem
- As a group, go over the results (use the transparency **Give It a Chance [T6]**).
  1. What is the probability of choosing a red tee? ($P(R)=5/10=1/2$)
     - o The event of red is 5 out of 10 possible tees
  2. What is the probability of choosing a blue tee? ($P(B)=2/10=1/5$)
     - o The event of blue is 2 out of 10 possible tees
  3. What is the probability of choosing a green tee *and* a blue tee, $P(G \cap B)$, in one draw? ($P(G \cap B) = 0$)
     - o It is *impossible* to draw a green *and* a blue tee at the same time
  4. What is the probability of choosing a yellow tee? ($P(Y)=0$)
     - o It is *impossible* to draw a yellow tee
  5. What color tee is the most likely to be chosen on a single draw? Explain. (Red is the most likely color to be drawn with theoretical probability because there are more red tees to be drawn.)
  6. Are the outcomes of a single draw *mutually exclusive*? Explain. (Yes, the outcomes are mutually exclusive; once a tee is drawn, no other tee can be an outcome.)

7. What is the probability of drawing a red tee *or* a green tee, $(P(R \cup G)$?
   $(P(R \cup G)) = 8/10 = 4/5$
   - As the events are mutually exclusive, the probabilities can be added, which gives $P(R) = 5/10 = 1/2$ or $P(G) = 3/10$, giving $P(R \cup G)) = 8/10 = 4/5$
7. What is the probability of drawing a tee that is *not* green?
   $(P(R \cup B)) = 7/10)$
   - This problem requires reasoning that is not given directly in the problem
   - If the tee is not green, it must be red or blue
   - The rules for mutually exclusive events apply also in terms of subtraction in the form of 10 tees minus the 3 green tees leaving 7 possible tees or 7/10 that are not green

Note to Instructor: There is an additional notation for "not" concepts but this activity stays with prior knowledge for mastery.

Goal 2: Develop integer concepts from concrete experiences.

2.1 Activity: Game of Ascent

The paraeducator will use concrete objects to define rules for integer addition.

Materials:
- Handout **Game of Ascent (H6)**
- Scissors or precut cards from the handout **Game of Ascent Cards (H7)**
- 2 paper clips per pair

2.1.1 Steps
- Have paraeducators complete the game on the handout **Game of Ascent (H6)** in pairs. Make sure the pairs carefully record their final scores.
- Ask pairs to complete the followup questions on the handout. Go over the answers as a group looking for patterns to create rules.
  - ▲ #1: Each player had numbers that were opposite.
  - ▲ #2: Answers will vary. There are opposites for each number in the original deck.
  - ▲ #3: As the rules stipulate that the player cannot draw all numbers, the maximum value is 15 as a player might choose all solid numbers and none of the outlined numbers.
  - ▲ #4: 8
  - ▲ #5: 3
  - ▲ #6: -3
  - ▲ #7: -4

2.2 Lecture: Developing Integer Concepts

**** Note to Instructor**:* Remind participants to use their Math Journals to take notes during lecture periods.

The Game of Ascent (The STEM Project, 1993-1998) introduces the concept of *integers* (use the transparency/handout **Integers [T7/H8]**).

Integers
Positive and negative whole numbers

Throughout the early elementary grades, students work with counting and whole numbers. The set of whole numbers includes zero. *Integers* are positive and negative whole numbers.

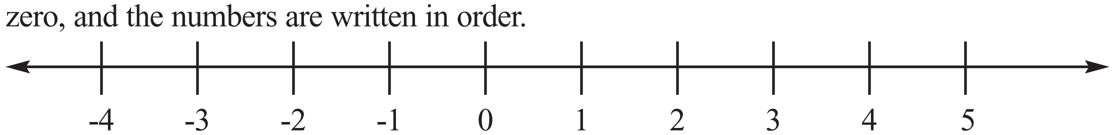
****Note to Instructor:*** Make sure the participants understand that the term *integer* does not include decimals or fractions. If we refer to positive and negative decimals and fractions, it may be confusing since many elementary school textbooks refer to those as signed numbers. In later grades, they are called rational numbers.

The game used a “ladder” rather than a traditional number line to assist students in understanding increase and decrease for these numbers. Participants should have realized that the outlined numbers were *negative* numbers and the solid numbers were *positive numbers*. *Negative* numbers implied a decrease in value; *positive* numbers implied an increase in value.

Adults often have an easier time with negative concepts than young students because adults comprehend bouncing a check or below zero or below sea level.

Sketch a number line on the transparency/handout **Integers (T7/H8).** Discuss that zero is the *origin*. Students tend to place zero in the middle of every number line. Explain that the placement of zero on a number line is arbitrary as long as its relationship to the positive and negative numbers is in the middle. We think of zero as neutral; no sign value is attached.

Positive numbers are placed on the number line in normal counting order where one comes directly after zero. Negative numbers are also in number order, but -1 is next to zero, and the numbers are written in order.



The most difficult element is size comparison. In general, moving to the right on a number line numbers always get greater. Moving to the left on a number line numbers always gets smaller. Paraeducators generally have little difficulty with positive number comparison or positive-to-negative comparison. However, they often struggle when comparing two negative numbers. Remind them that the placement on the number line should help them reason out the comparison. This makes the use of the number line vital to success with integers.

2.3 Discussion: Comparing Integers

The paraeducator will use the number line to compare integer and signed number values.

2.3.1 Steps

- Have participants individually complete the comparison problems on the transparency/handout **Integers (T7/H8).**
- May need to quickly review > < symbols from the K-4 Academy.

Note to Instructor: Make sure to use the proper terms for negative numbers. The number "-2" should be read as "negative two" not "minus two." Minus is an operation that will be discussed in upcoming goals.

- Go over comparisons as a group to check for understanding.
  - ▲ a) 6 __<__ 8
    - o This should be an easy comparison of positive numbers.
  - ▲ b) 0 __>__ -2
    - o Even though zero is not positive or negative, it rests to the right of -2, making it the larger number.
  - ▲ c) -2 __>__ -4
    - o This might be challenging for some because usually 4 is greater than 2. Remind participants that the number to the right is always larger; success here depends on a good understanding of the number line.
  - ▲ d) -6 __<__ 3
    - o Most participants manage positive and negative comparisons with ease as any positive number is greater than any negative number. Lack of attention to detail usually causes errors on this problem.
  - ▲ e) -5 __>__ -7
    - o This is similar to c) above. This is challenging because the numbers were not listed on the class example number line. Participants must be able to generalize the number line in order to use it effectively to predict outcomes.
  - ▲ f) -1/2 ___>___-2
    - o -1/2 is a signed number (not an integer). Some discussion may need to occur about the placement of -1/2. The negative should signal which side of zero. Ask participants to figure out where $\frac{1}{2}$ would be and then mirror that placement on the other side of zero. Negative one half should be between 0 and -1.

Goal 3: Develop rules for integer addition and subtraction from concrete experiences.

3.1 Discussion: Recounting the Game of Ascent

The paraeducator will connect concrete experiences to general integer operation rules.

3.1.1 Steps

- Each time a card was drawn in the Game of Ascent, an associated addition problem could be created, and addition means "and."
- Ask participants to look at the Game of Ascent ladder and act out the following scenarios. Make sure to start at zero for each scenario. As a group, write an associated addition problem for each scenario.

  ***Note to Instructor 1**: In mathematics, we do not have to put the "positive" symbol on positive numbers as we do for negative numbers. It is assumed that the number is positive if it does not have a symbol.

  ***Note to Instructor 2**: The notes below use the words *up* and *down*; transition as quickly as possible to right (for up) and left (for down) to ease into the use of the traditional number line.

  - ▲ Solid 2 and solid 3: (says start at 2 and go up 3; ends up at 5 on the ladder)
    - o 2 + 3 = 5
    - o This is an expected result from early elementary grades
  - ▲ Outlined 1 and outlined 3: (says start at -1 and go down 3; ends up at negative 4 on the ladder)
    - o -1 + -3 = -4
  - ▲ Outlined 3 and outlined 4: (says start at -3 and go down 4; ends up at negative 7 on the ladder)
    - o -3 + -4 = -7
- Ask participants if they can describe what happened in the second and third scenarios; try to generate a rule for all three examples for addition of integers.
  - ▲ To add like signs, either both positive or both negative, the numbers are added and the sign is kept (add this rule to the transparency/handout **Integers [T7/H8]**)
- Ask participants to look at the Game of Ascent ladder and act out the following scenarios using the transparency **Integer Practice (T8).** Make sure to start at zero for each scenario. As a group, write an associated addition problem for each scenario. Participants should add examples to the notes.
  - ▲ Solid 2 and outlined 3: (says start at 2 and go down 3; ends up at -1 on the ladder)
    - o 2 + -3 = -1
  - ▲ Outlined 4 and solid 5: (says start at -4 and go up 5; ends up at 1 on the ladder)
    - o -4 + 5 = 1
  - ▲ Outlined 1 and solid 3: (says start at -1 and go up 3; ends up at 2 on the ladder)
    - o -1 + 3 = 2

- Ask participants if they can describe what happened in these examples. Try to generate a rule for all three examples. This will be the most challenging for participants to see.
  - ▲ To add unlike signs, we think subtraction and keep the sign of the "larger" number (add this rule to the transparency/handout **Integers [T7/H8]**)

***Note to Instructor*:** Be very careful with the term *larger* as it actually refers to the absolute value of the number (some students may already know this concept).

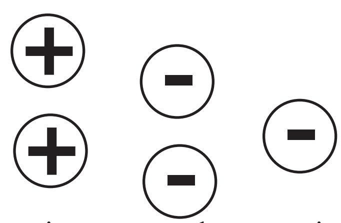
- Clarify the rule by adding the phrase "keep the sign of the number that is more positive or more negative" (this will be clarified in the following lecture)

3.2 Lecture: Adding Integers

***Note to Instructor**:* Remind the participants to take notes throughout lecture periods.

To clarify the last rule developed above, use the charged particle model (idea of charges on particles) or the chip model as shown below so that participants have another model on which to build their knowledge.

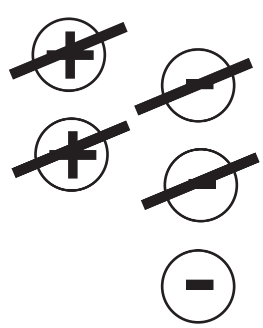
Use the first example of 2 + -3, which says "two positives plus three negatives."



When adding integers, an important rule comes into play (use the transparency/handout **Integers [T7/H8]**).

Adding a positive and negative of the same number will cancel to be zero. Demonstrate this with something simple on the number line like -1 + 1 = 0. We would say that 1 and -1 are *opposites.*

In the charged particle or chip model, a positive will always cancel a negative when adding integers.



We can see that there are more negatives than positives, so the final answer must be negative. Participants may need to draw particle or chip models until they can internalize the rules. Chip models may also be used for same-sign problems if students lack prior understanding.

- For 2 + -3 , they are opposite in sign, so we think subtraction and keep the sign as negative because there are more negatives.
- For -4 + 5 , they are opposite in sign, so we think subtraction and keep the positive sign because there are more positives.
- For -1 + 3, they are opposite in sign, so we think subtraction and keep the sign of positive because there are more positives.

3.3 Activity: Game of Ascent II

The paraeducator will use concrete objects to define rules for integer subtraction.

Materials:
- Handout **Game of Ascent (H6)**
- Handout **Game of Ascent II (H9)**
- 2 paper clips per pair
- 1 coin per pair

3.3.1 Steps
- Assign heads to be "+" and tails to be "-" or "reverse" to alleviate any questions.
- Complete the handout **Game of Ascent II (H9)** in pairs. Make sure the pairs carefully record their final scores.
- Ask pairs to complete the followup questions for the handout **Game of Ascent II (H9)**. Participants may struggle with #3 and #4. Go over the answers as a group looking for patterns to create rules.

***Note to Instructor**: Parentheses are used in some of these problems to ease confusion of too many signs. It does not change the problem.*

- ▲ #1: No, there appears to be no pattern. The operator is due to chance
- ▲ #2: Game of Ascent II is based more on luck due to the element of chance
- ▲ #3: -4
- ▲ #4: -3+(-2) + (-1)+2=-4

3.4 Discussion: Recounting the Game of Ascent II

The paraeducator will connect concrete experiences to general integer operation rules.

3.4.1 Steps
- This game threw in a twist, the coin operator. The operator had the power to change the value of the card drawn to its opposite, which changed the strategy of the game.

- This game demonstrated subtraction rules for integers. Commonly, intermediate and middle -grade students struggle the most with subtraction compared to the other three operations.
- An important development for intermediate and middle school learners is to see subtraction as *opposite* of addition, just as we would see 1 and -1 as *opposites.*
- Ask participants to look at the Game of Ascent ladder and act out the following scenarios using the transparency **Integer Practice (T8).** Make sure to start at zero for each scenario. As a group, write an associated subtraction problem for each scenario. Participants should add examples to the notes.
    - ▲ Outlined 3, heads, outlined 4, tails: (says start at -3 and do the opposite of going down 4, which means to go up 4; ends at 1)
        - o -3 – (-4) = 1
    - ▲ Solid 2, heads, outlined 5, tails: (says starts at 2 and do the opposite of going down 5, which means to go up 5; ends at 7)
        - o 2 – (-5) = 7
- Ask participants if they can describe what happened in these examples. Try to generate a rule. This will be challenging for participants; they may need some guidance on the first subtraction examples.
    - ▲ Each time, the first number was not affected
    - ▲ The second number changed to its opposite, then the addition rules were used (most of the paraeducators will not see this yet – can save this until the end of the discussion)
- Ask participants to look at the Game of Ascent ladder and act out the following scenarios. Make sure to start at zero for each scenario. As a group, write an associated subtraction problem for each scenario.
    - ▲ Solid 1, heads, solid 3, tails: (says start at 1 and do the opposite of going up 3, which means to go down 3; ends at -2)
        - o 1 – 3 = -2
    - ▲ Outlined 4, heads, solid 4, tails: (says start at -4 and do the opposite of going up 4, which means to go down 4; ends at -8)
        - o -4 – 4 = -8

***Note to Instructor*:** Many participants will say 0 for the answer because they appear to be opposites. If this was addition, the answer would be 0. However, the operator changes the sign.

- Ask participants if they can describe what happened in these examples. Try to generate a rule. This will be challenging for participants; they may need some guidance on the first subtraction examples.
    - ▲ Similar generalizations, if possible, should come here from the above examples
    - ▲ It is likely that participants will not see any generalizations
    - ▲ This will be addressed in the following lecture

3.5 Lecture: Subtracting Integers

***Note to Instructor*:** Remind participants to take notes throughout lecture periods.

It is not unusual to find that none of the participants could verbalize what they saw happening when they created subtraction problems. The examples appear to have no pattern because they look different. Paraeducators feel overwhelmed with subtraction of integers because they see more patterns and rules to memorize.

The charged particle or chip model might be used here, but it can be difficult to comprehend. To demonstrate 1 – 3, this says to take away 3 positives from 1 positive; this does not seem possible. The problem -3 – (-4) says to take away 4 negatives from 3 negatives; this also does not seem possible. The last example has a positive answer, which seems to go against all logic from prior concrete experiences with subtraction from grades K-4.

In keeping with the vocabulary developed throughout Goal 3, subtraction must be seen as the opposite of addition. In mathematics, we begin to realize that all subtraction may be rewritten as addition. This means that the only "rules" that have to be memorized are the two rules for addition.

The easiest example is one used in a prior lesson to show adding opposites. For the problem 1 + (-1), the sum was zero. Ask participants which basic subtraction problem would also give an answer of zero. The problem is 1-1. This means that these two problems are equal.

Subtraction of integers (or signed numbers) may be changed to addition. It is important to note that it does not always have to be changed such as in a basic problem of 4 – 3; we know this as 1. For all other problems, there must be a two-step process: "add the opposite" (refer to transparency/handout **Integers [T7/H8]**). This requires seeing each problem in three pieces:

$$-3 - (-4) \rightarrow -3 \quad - \quad (-4)$$

Following the previous rule, we must (a) add and (b) change the second number to the opposite.

$$-3 \quad - \quad (-4)$$
$$-3 \quad + \quad 4$$

This returns back to the rules of addition for unlike signs, which say think subtraction and keep the sign of the more positive or the more negative.

$$-3 + 4 = 1$$

It is important for each problem to say the phrase "add the opposite" as each step is being performed. There are many common phrases and processes for working integer problems; use any that are appropriate.

$$2 \quad - \quad (-5)$$
$$2 \quad + \quad 5 = 7$$

$$1 \quad - \quad 3$$
$$1 \quad + \quad (-3) = -2$$

$$-4 \quad - \quad 4$$
$$-4 \quad + \quad (-4) = -8$$

The last two examples are difficult because it is hard to remember that the 3 and the 4 are initially positive numbers that need to change to the opposite numbers in sign.

***Note to Instructor**: A discussion may arise about which form is more appropriate in an example such as 3 – 5 or 3 + (-5). In truth, it does not matter. There are times when one is easier to manipulate than another. As participants read these problems differently, it is difficult to say what is appropriate. The most important thing to remember is that they are equivalent expressions.*

Goal 4: Develop rules for integer multiplication and division from analyzing patterns.

4.1 Discussion: Multiplication of Integers

The paraeducator will use past knowledge of addition and multiplication to create rules for multiplication of integers.

4.1.1 Steps

- Remind participants that the rules for multiplication and division of integers come from analyzing and developing patterns from early elementary grades.
- Return to basic addition and multiplication links (use the transparency/ handout **Integers [T7/H8]**).
  - ▲ Ask participants to write a multiplication problem for 2 + 2 + 2 + 2 (2 x 4)
  - ▲ Ask participants to compare the answers (both answers check to be 8)
- Present the problem (-3) + (-3) + (-3) + (-3)
  - ▲ Ask participants to write are related multiplication problem (-3 x 4)
- Have participants use the rules from Goal 3 to evaluate the addition problem.
  - ▲ The answer for adding like signs is -12
  - ▲ This means that -3 x 4 must equal -12
- Present the problem (-5) + (-5) + (-5)
  - ▲ Ask participants to write a related multiplication problem (-5 x 3)
- Have participants use the rules from Goal 3 to evaluate the addition problem (-5) + (-5) + (-5)
  - ▲ The answer for adding like signs is -15
  - ▲ This means that –5 x 3 must equal -15
- Most participants work well with these examples because they can link them back to addition concepts.
  - ▲ Point out that if the order were reversed, such as 3 x -5, writing the addition problem would not be possible unless the order was switched
  - ▲ Remind participants that this is the reason for focusing on the connections between addition and multiplication in early elementary
- Ask for any rule generalizations that could be drawn from these examples.

***Note to Instructor**:* If no generalizations can be drawn, do not force the issue.

- ▲ Generalization: A positive times a negative is negative
- Present the problem (–4) x (–4)
  - ▲ Ask participants to write a related addition problem (this is not possible in this form; it is impossible to write something negative four times)
  - ▲ Point out that sometimes models cannot be applied easily or at all to every situation. Sometimes the problem must be creatively altered for the model to work
  - ▲ Explain that there must be a way to create rules for these types of problems

4.2 Activity: Operation Integer

The paraeducator will use patterns to define rules for multiplication and division of integers.

Materials:

- Transparency/handout **Operation Integer (T9/H10)**

4.2.1 Steps

- In pairs, have participants complete the handout **Operation Integer (T9/H10).**
- Encourage them to look for patterns and not to focus on memorizing rules; these patterns come from early elementary understanding of multiplication with the added issue of the signs.
- Go over the handout to make sure that all patterns were followed (answers are already filled in on the transparency **Operation Integer [T9]**).
  - ▲ For charts #1 and #2, make sure to discuss methods for filling in missing information
    - o One column was a constant
    - o The other column was decreasing by one
    - o When zero was passed, negative numbers were included
    - o Basic multiplication strategies of skip counting or using basic facts could have been used
  - ▲ If participants do not have an answer for #7, keep it covered until after the followup lecture

4.3 Lecture: Making Rules

***Note to Instructor**:* Remind participants to take notes throughout lecture periods.

The prior activity presented many exercises for patterns. The key is to be able to generalize those patterns into workable rules for multiplication and division. While this activity only dealt with multiplication, remind participants that multiplication and division are linked and in integers, they share identical rules. The rules that are developed for multiplication also apply to division.

The easiest way to organize this information is to look at outcomes from the activity and the discussion. Use the transparency/handout **Integers (T7/H8)** to record the information.

Start with outcomes that were positive in value from the product *of two* integers. These may be generalized to *like signs*, either both positive or both negative, producing a positive result. Add this to the transparency/handout **Integers (T7/H8).** The positive factors portion of this rule matches what is known about regular multiplication.

Look for outcomes that were negative in value from the product of two integers. These may be generalized to *unlike signs* (where one integer was negative) producing a negative result.

These examples are a good place to start but do not truly cover the rules for integer multiplication and division. Look at the problem that had more than two integers (problem #7). Go through the same exercise of looking for positive and negative outcomes.

To generalize the rules, participants must look for broader patterns. It is easy to get caught up in placement of the negative sign such as -3 x 4 vs. 4 x -3. Remind participants that placement in multiplication has nothing to do with the final product (Commutative Law of Multiplication).

The rule for multiplication and division focuses on two issues: the type of sign and the number of signs, specifically negative signs.

An even number of negative signs creates a positive product.

Participants should have been able to see this from the like sign examples. The product of every two negatives makes a positive. This is similar to the adage of "two wrongs making a right."

An odd number of negative signs creates a negative product.

Participants may need to go back and analyze this information. Add these extended rules to the transparency/handout **Integers (T7/H8).** Remind participants that these rules also apply directly to division.

Complete the brief division practice as a group. The rules are identical to those of multiplication.

Answers:
1. -5 (one negative means the answer is negative)
2. 3 (even number of negatives gives positive answer)
3. -4 (one negative means answer is negative)
4. 6 (positives give positive – standard elementary division rules)

Goal 5: Explore the coordinate graph system.

5.1 Lecture: Coordinate Graph System

Note to Instructor: Remind participants to take notes throughout lecture periods.

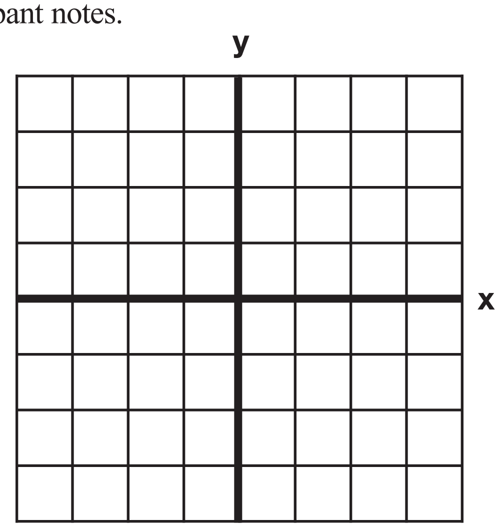
Opening up the real number system to include integers is a major step in developing many other mathematical concepts. This allows for the introduction of a major concept of graphing. The coordinate graph system is introduced here as a practice with integer concepts. This concept will be used later in this Academy to show relationships between concepts and to display data.

Use the transparency/handout **Coordinate Graphing (T10/H11)** throughout this lecture.

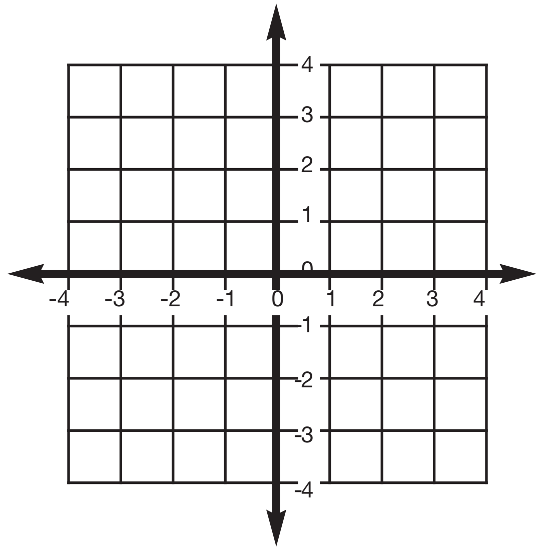
The *coordinate graph system*, also known as the *Cartesian coordinate system*, functions on the placement of *ordered pairs*. To have ordered pairs, there must be a recognized system of naming.

A *coordinate graph system* includes a set of *axes* (plural for axis). In two dimensions, each axis has a name: *x* is the *horizontal axis,* and *y* is the *vertical axis*. Although these are often assumed, it is important to always label the axes to avoid confusion.

Note to Instructor: The axes here should have arrows on the ends to show a continuous line. For ease of graphics, they are not shown here but should be added for participant notes.



The axes should be thought of as two number lines. The x axis reads from left to right like a traditional number line. The y axis reads from bottom to top similar to the ascent ladder. The labeling process starts at the center, or *origin.*



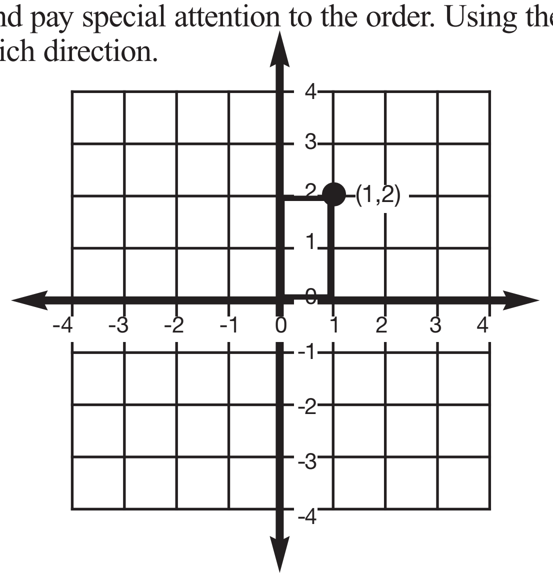
It is important to note that the lines, not the spaces, are numbered.

A *coordinate*, also called an *ordered pair*, is a location. Most participants will be familiar with a coordinate system if they have ever located a city on a map. The city location is commonly given with a letter and a number. A given city is at the intersection of those two directions. Most people run their fingers from both directions until they touch. Graphing in a coordinate system is very similar.

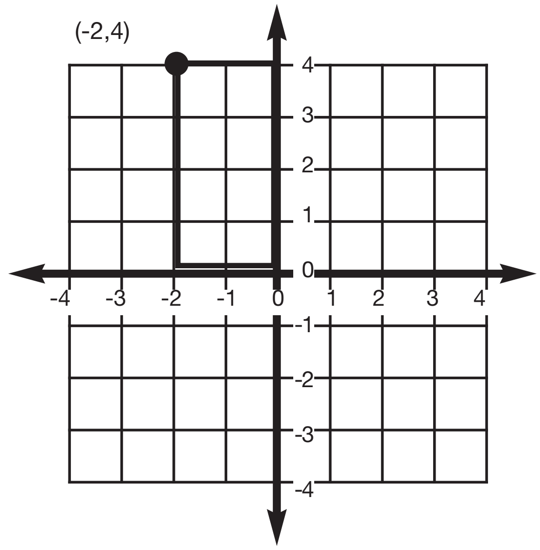
We know from personal experience that order matters when giving directions to our homes, for example. In other words, it makes a difference whether someone turns left first, then right or the opposite. A coordinate is named as *(x, y),* implying that the direction of x is considered first, then the direction of y. The parentheses denote an ordered pair.

The *origin* has a coordinate of (0,0). This implies no movement in the x or y direction.

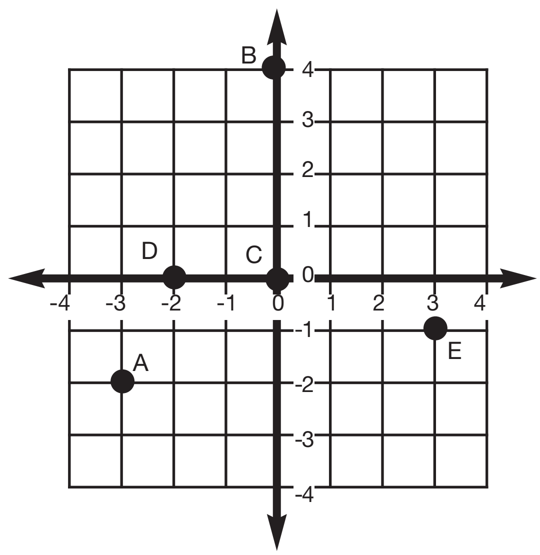
The coordinate (1, 2) says from 0, go 1 to the right on the x axis (as 1 is positive) and then 2 up on the y axis (as 2 is positive). This movement results in the point shown below. It is important to use proper notation and pay special attention to the order. Using the term *over* to describe movement does not tell which direction.



The point (-2, 4) says go left on the x axis (as 2 is negative) and then go up 4 on the y axis (as 4 is positive).



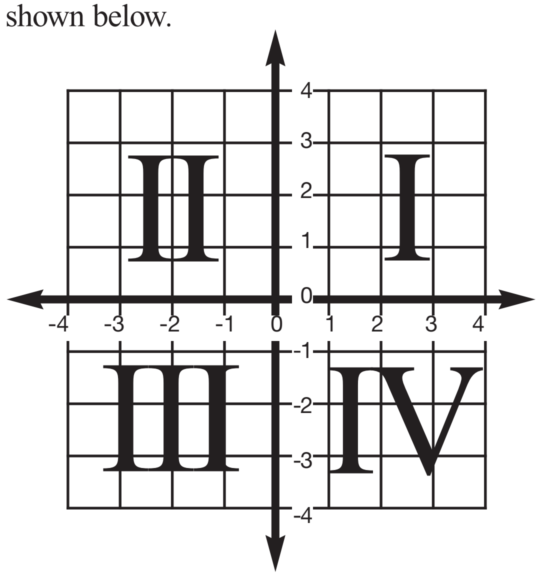
As a group, do some quick practice of naming the following points. Use the transparency/handout **Naming Coordinates (T11/H12).**



Answers:

A: (-3, -2)
B: (0, 4)
C: (0,0)
D: (-2, 0)
E: (3, -1)

While naming points is an important skill, it is also important to generalize patterns that occur on a graph. Participants should have noticed that the axes divide the graph into four *quadrants*. Each quadrant has a specific pattern that each coordinate possesses in terms of positives and negatives. Label the quadrants as shown below.



This may best introduced as participants share the patterns that they see.

All coordinates in Quadrant I are (+, +).
All coordinates in Quadrant II are (-, +).
All coordinates in Quadrant III are (-, -).
All coordinates in Quadrant IV are (+, -)

It is important that participants see that each quadrant is unique and the coordinates in those quadrants share those patterns. Participants should be able to describe how to move a point from one quadrant to another by changing a sign. For example, to move the point (-1, 2) into quadrant III, the sign of the *y coordinate* needs to change to a –2, creating the point (-1, -2). Practicing this skill will give participants a stronger base for working with coordinates and later with graphing concepts.

5.2 Activity: "Sink My Ship" (also known as Battleship)

The paraeducator will develop coordinate graphing skills through concrete experiences.

Materials:

- Handout **Centimeter Grid Paper (H13)** (2 per participant)

5.2.1 Steps

- Have each participant create a coordinate system (10 x 10 – x and y axes should go from –10 to 10) on both sheets of the grid paper with both axes labeled x and y.
- On one sheet, have participants draw a line that represents a "ship" on the grid paper.

- ▲ The line must have three coordinate points
- ▲ The line can have any orientation on the paper
- ▲ Have participants label each coordinate of the line.

****Note to Instructor***: May need to demonstrate the "ship."

- In pairs, have participant take turns guessing at the location of the partner's "ship" by guessing a coordinate.
  - ▲ For each guess, the partner will say "hit" or "miss"
    - o A "miss" represents a guess that is not on the "ship" or line
    - o A "hit" is one of the coordinates on the line
    - o The guesser should record each guess on his or her blank grid
- To sink a "ship," three hits are required.
- Once a partner sinks a ship, he or she must show the suspected location on the grid paper so the partner can check if his/her ship has been sunk
- The winning participant sinks a partner's ship in the fewest number of guesses.
- This game is an effective way to practice coordinates and making inferences based on data.

6.1 Assignment #1: Probability and Integers

****Note to Instructor***: Decide how long class members will have to complete their assignments so that you have time to grade, record grades and turn in materials from this Academy in a timely manner. If paraeducators are taking the Academy for credit, there will be a time limit based upon the grading period at the attending institution. You will also have to decide how you would like attendees to turn in their assignments to you. For example, they can be mailed or you can make whatever arrangements work for you and your class. **You are strongly encouraged to be firm about the completion date and may need to make some effort to follow up on attendees and their progress. Refer to the Grading Rubric handout (GR) for details on grading.**

Distribute the handout **Assignment #1: Probability and Integers (H14).** Read the instructions and answer questions regarding completion of the assignment. Provide the class with a date for completion and explain your process for handing the assignment in. **To assist in grading the assignment, answers to the questions are listed under each question. Please ensure that the answers are not released to the students before they complete the assignment.**

The assignment is worth 130 points. The focus of this assignment is to practice probability skills and master integer operations. There are two parts to the assignment.

***Part 1*: (65 points)**
Use a traditional die (single for dice) for the following questions. Make sure to reduce fractions where appropriate.

1. List the sample space for a single die.

   Answer:
   $S=\{1, 2, 3, 4, 5, 6\}$

2. What is the probability of rolling a 3 on a single roll of the die?

   Answer:
   $P(3)=1/6$

3. What is the probability of rolling an even number on a single roll of the die?

   Answer:
   $P(2\cup 4\cup 6)=3/6=1/2$

   Note: The fraction should be reduced. There should be some evidence of how the 3 out of 6 evolved.

4. Explain your method of reasoning for #3 above.

   Answer:
   These events are mutually exclusive so their individual probabilities may be added together. The opportunity to roll an even number is 2, 4, 6. Since each is 1/6, the total is 3/6=1/2.

5. What is the probability of *not* rolling a 4 *or* 5?

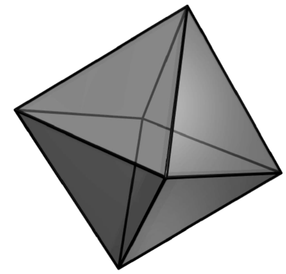
   Answer:
   $P(1\cup 2\cup 3\cup 6)=4/6=2/3$

6. Explain your method of reasoning for #5 above.

   Answer:
   The idea of "not" means that there are other options. Rolling a 4 or 5 is a probability of 2/6. As the events are mutually exclusive, the probability may be subtracted. The probably of rolling something other than a 4 or 5 is 6/6–2/6=4/6=2/3.

7. Answer the following imagining an eight-sided die with numbers 1-8.
   - List the sample space for a single die.

     Answer:
     S={1, 2, 3, 4, 5, 6, 7, 8}



   - What is the probability of rolling a 2 on a single roll of the die?

     Answer:
     $P(2)=1/8$

   - Are you more likely to roll a 2 on a 6-sided or 8-sided die? Explain your answer.

     Answer: If these are changed into percents (not required), $1/6=16\frac{2}{3}\%$ and $1/8=12\frac{1}{2}\%$, it is easy to see that you are more likely to roll a 2 on the six-sided die. Without percents, the reasoning can be made that there are more options on an eight-sided die to roll something other than a 2. It should be easier on a die with fewer options.

Part 2: **(65 points)**

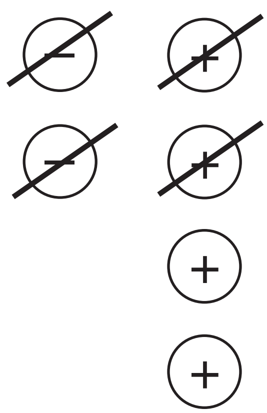
Evaluate the following problems WITHOUT a calculator. Show enough work to support your answer.

1. 4+(-7)=-3
2. -3–(-7)=4
3. (-9) • (-6)=54
4. 35÷(-5)=-7
5. (-2) + (-5) = -7
6. 3 • (-4) = -12
7. -1 – 4 = -5
8. -1 + 2 – 3 = -2
9. (-1) • (-2) • (4) = 8
10. Draw a chip (or charged particle model) or number line to demonstrate the following

problem. Explain your diagram. Include information on the resulting sign choice.

(-2) + 4

Answer: A number line model may also be used. This chip model shows two negatives and four positives. When adding, a positive and a negative cancel to be zero. All of the negatives cancel leaving two positives, or 2 as the answer.



Module B Handouts

Probability Review

Probability:
- The chance of an event occurring
- Value between 0 and 1
- Requires the use of fractions
- Determined by the number of possible outcomes

Equally likely:
There is equal *probability (chance)* for an outcome.

Fair:
Object or event has options that are equally likely.

Likelihood:
How probable or likely it is that something will happen.

Likelihood is also known as **probability**.

Impossible: Event will never occur.

Certain: Event will absolutely occur.

Using Probability and Prediction

- Predict behaviors
- Explain past occurrences from patterns or data

- Real-life examples:
  - ▲ The gaming (casino) industry relies on theoretical probability to make a profit
  - ▲ Weather forecasters
  - ▲ Medical industry
  - ▲ Insurance companies

Naming Probability

For a coin, heads should show up 50% or 1/2 of the time.

$P(H) = 1/2$ says the probability (P) of getting a heads (H) is 1/2

Determining probability:

1. Sample Space (S)

   Set of all possible outcomes

   - Coin: $S = \{H, T\}$
     S = sample space
     H = heads
     T = tails
     { } imply a set or grouping

   - This becomes the **denominator** of the fraction to represent the probability

2. Event

   Any subset or subgroup of the sample space

   - The event becomes the numerator

   - For the coin, heads (H) is one of two possible events

3. Determine the probability

   a) Theoretical probability:

      - A mathematical computation where A is the defined event; can be a single event or multiple events

$$\frac{\text{Number of elements in A}}{\text{Number of elements of sample space (S)}} = \frac{n(A)}{n(S)}$$

   b) Experimental probability:

      - Same calculation but the data come from an experiment

Making Heads or Tails of Probability

▲ What is the probability of getting a heads or tails on one flip?

▲ What is the probability of getting a star on a coin?

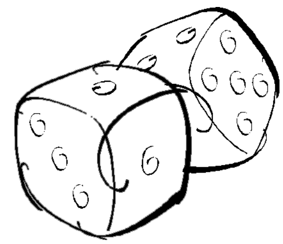
▲ What is the probability of getting a heads and a tails on one flip?

Mutually Exclusive

When event A occurs, event B cannot occur.

- For multiple events, probabilities for mutually exclusive events may be added together to determine the final probability.

Give It a Chance



A bag contains 3 green golf tees, 2 blue tees, and 5 red tees. Use this information to answer the following questions. Use proper probability notation in your answers.

1. What is the probability of choosing a red tee?

2. What is the probability of choosing a blue tee?

3. What is the probability of choosing a green tee *and* a blue tee, $P(G \cap B)$, in one draw?

4. What is the probability of choosing a yellow tee?

5. What color tee is the *most* likely to be chosen on a single draw? Explain.

6. Are the outcomes of a single draw *mutually exclusive*? Explain.

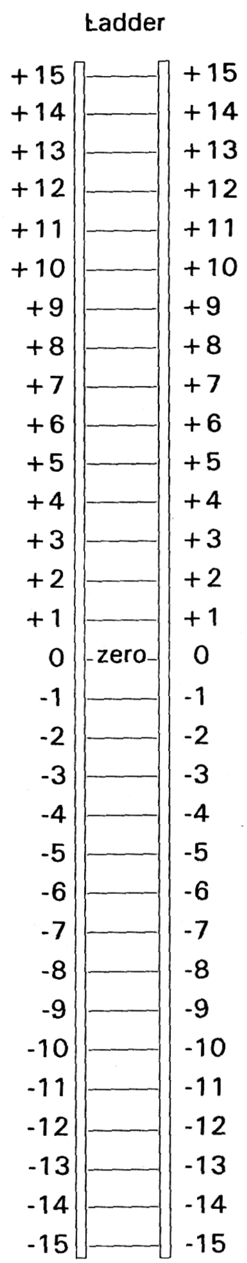
7. What is the probability of drawing a red tee *or* a green tee, $P(R \cup G)$?

8. What is the probability of drawing a tee that is *not* green?

The Game of Ascent

Winning at games often requires a combination of luck, intuition, and strategy. Many strategies rely on the player's ability to recognize number patterns, and to apply various mathematical skills.

How about you? Can you use your number sense to develop a successful strategy for the game of Ascent? Read the directions below to learn how to play the game.



The Game of Ascent

- Ascent is a game for 2 players. At the start, each player places a marker on the 0 rung of the ladder at the left.
- The card deck is shuffled and one player starts the game by drawing a single card from the deck. Players alternate turns starting each game.
- If an outlined number is drawn, a player moves **down** the ladder the number of rungs shown on the card. For example, since an outlined three represents -3, a player drawing that card must move down three rungs on the ladder.
- If a black number is drawn, the player moves **up** the ladder. Drawing a black 5 (+5) would move a player 5 rungs up the ladder.
- The player who begins the game draws a card and moves to the appropriate place on the ladder. That player may then keep drawing cards or may stop and "pass" the draw to the opposing player at any time. After drawing once, the opposing player may pass the draw back to the beginning player or continue to draw more cards.
- **A player must draw at least one card each turn, and the beginning player is not permitted to draw all the cards.**
- Play continues until all the cards in the deck have been drawn. The winner of Ascent is the player who reaches the highest position on the ladder at the game's end.

Game of Ascent Followup

1. What pattern did you notice for each game?

2. What do you think caused this outcome?

3. What is the greatest possible score a single player can obtain in a game? Explain.

4. Which was higher on the game ladder, 8 or 5?

5. Which was higher on the game ladder, 3 or -2?

6. Which was lower on the game ladder, -3 or 0?

7. Which was lower on the game ladder, -2 or -4?

Game of Ascent Cards

5	**4**	**3**
2	**1**	0
1	2	3
4	5	

Integers

Integers

Positive and negative whole numbers.

Comparing Integers

Use > or < to compare the following values.

a) 6 _____ 8

b) 0 _____ -2

c) -2 _____ -4

d) -6 _____ 3

e) -5 _____ -7

f) -1/2 ______ -2

Adding Integers

To add like signs,

To add unlike signs,

Making zero

Adding a positive and negative (opposites) of the same number will cancel to be zero

Chip or charged particle model: 2 + (-3)

Subtracting Integers:

-3 – (-4) → -3 – (-4)

Steps:

1.

2.

2 – (-5)

1 – 3

- 4 – 4

Multiplying Integers:

Link to addition: 2 + 2 + 2 + 2

(-3) + (-3) + (-3) + (-3)

(-5) + (-5) + (-5)

(-4) x (-4)

Generalization:

Positive product:

Negative product:

Dividing Integers:

Same rules as for multiplication of integers.

1. -10/2
2. -15/-5
3. 16/-4
4. 60/10

The Game of Ascent II

Now we'll add a new element to the game of Ascent called the Operator. The Operator is a coin you'll flip after drawing a card. On one side of the Operator is a (-) symbol. This symbol reverses the direction of your move. For example: suppose you draw an outlined 5 and flip the Operator and it lands on (-). Then instead of moving 5 units down, you move 5 units up.

On the other side of the Operator is a + sign. This sign will not affect the direction given on the card. If you draw an outlined 3, and the Operator lands on + , you must move down 3 units, as you normally would.

Ready? Roll up your sleeves and try your luck at Ascent II.

Game of Ascent II Followup

1. Is the pattern the same for Ascent II as for Ascent I? Explain.

2. Which game relies more on luck, Ascent I or II?

The equation $2 + (-4) - (-1) + 0 = -1$ represents the following scenario:

Draws a solid 2, then flips heads (+)
Draws an outlined 4, then flips heads (+)
Draws an outlined 1, then flips tails (-)
Draws a 0, then flips heads (+)

3. Use the ladder to act out the following scenario to solve the equation:
   Draws an outlined 3, then flips heads
   Draws an outlined 2, then flips heads
   Draws a solid 1, then flips tails
   Draws a solid 2, then flips heads

4. Write a numerical expression to represent the problem.

Operation Integer

Agent "Math"ew was assigned the duty of cracking the Integer Code. He only remembers his basic multiplication rules from elementary school. He must crack the code to save the world. The future of mathematics depends on it. Use patterns to complete each set.

1.

3	x	3	=	9
3	x	2	=	
3	x	1	=	
3	x	0	=	
3	x		=	
	x		=	-6
	x		=	-9
	x		=	

2.

4	x	4	=	16
3	x	4	=	12
	x	4	=	8
	x	4	=	
	x	4	=	
	x	4	=	
	x		=	
	x		=	

3. Write a rule for multiplying a positive and negative number.

4.

-2	x	2	=	-4
-2	x	1	=	-2
-2	x	0	=	
-2	x		=	
-2	x		=	4
	x		=	6
	x		=	
	x		=	

5.

3	x	-5	=	
	x	-5	=	-10
	x	-5	=	-5
	x		=	
	x		=	
	x		=	
	x		=	
	x		=	

6. **Write a rule for multiplying two negative numbers.**

7. 2x2x2x2=
2x2x2x(-2)=
2x2x(-2)x(-2)=
2x(-2)x(-2)x(-2)=
(-2)x(-2)x(-2)x(-2)=

8. **Write a generalization that you see from #7.**

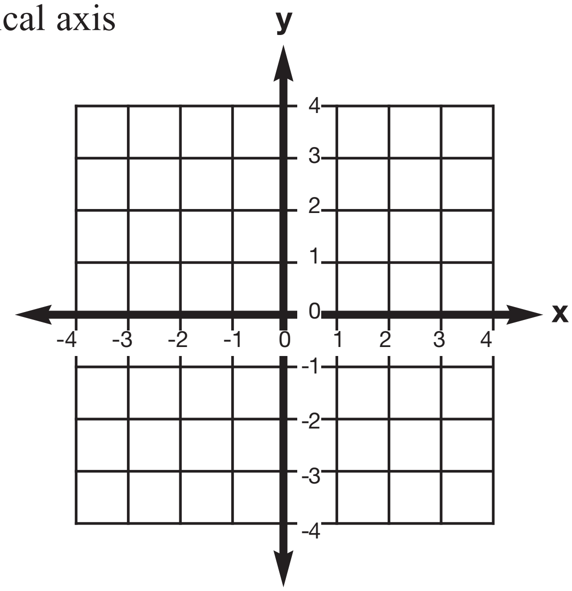
Coordinate Graphing

***Cartesian coordinates or coordinate graphing*:**

Requires ordered pairs and axes.

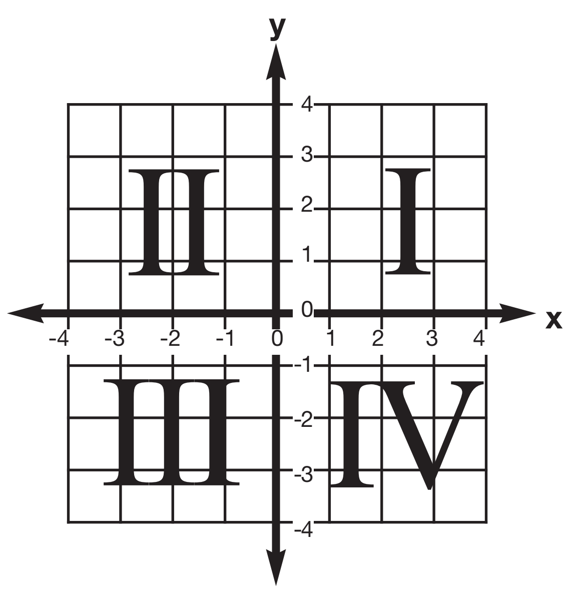
x axis: horizontal axis

y axis: vertical axis

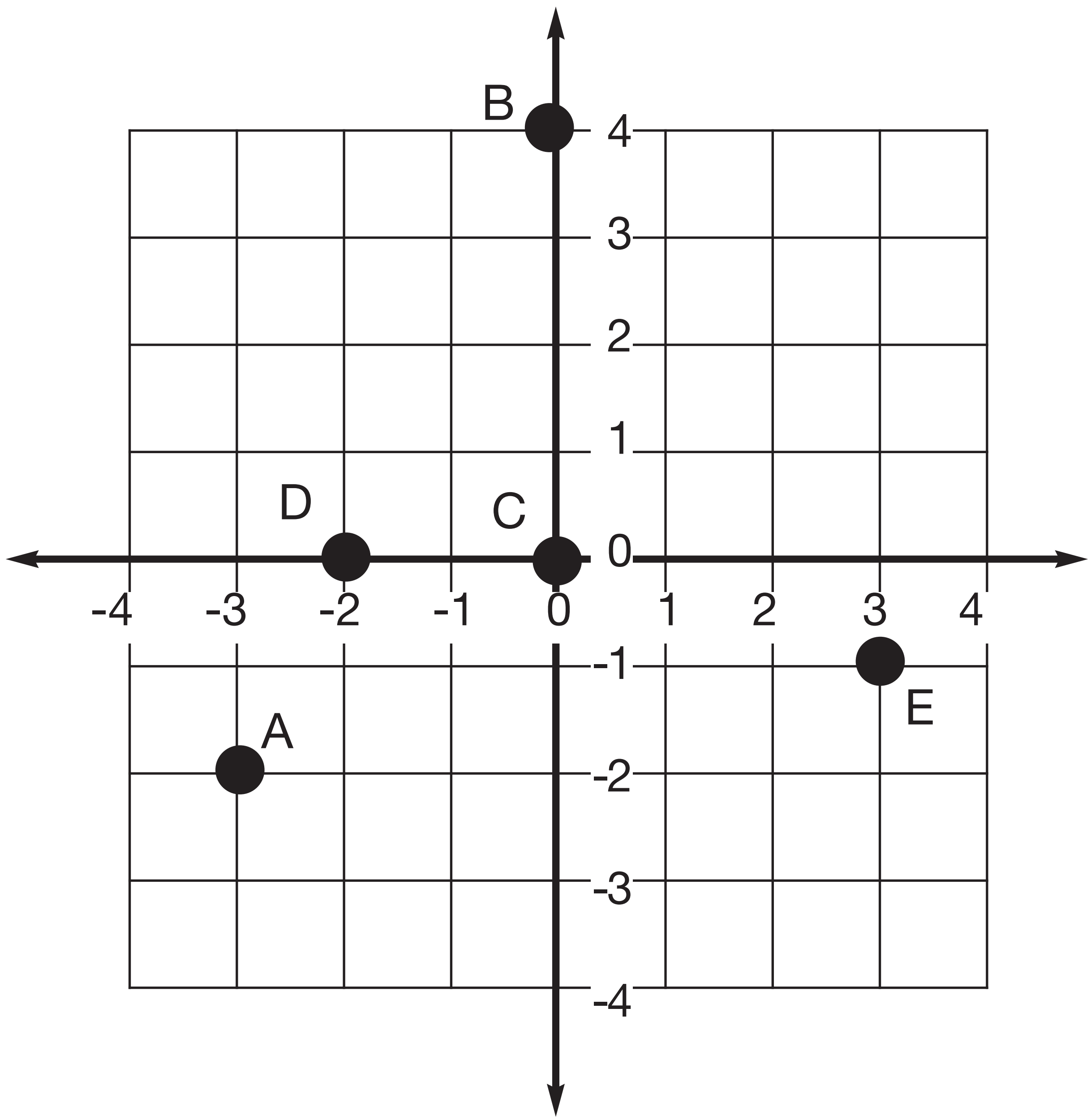


Coordinate or ordered pair: location in the form (x, y)

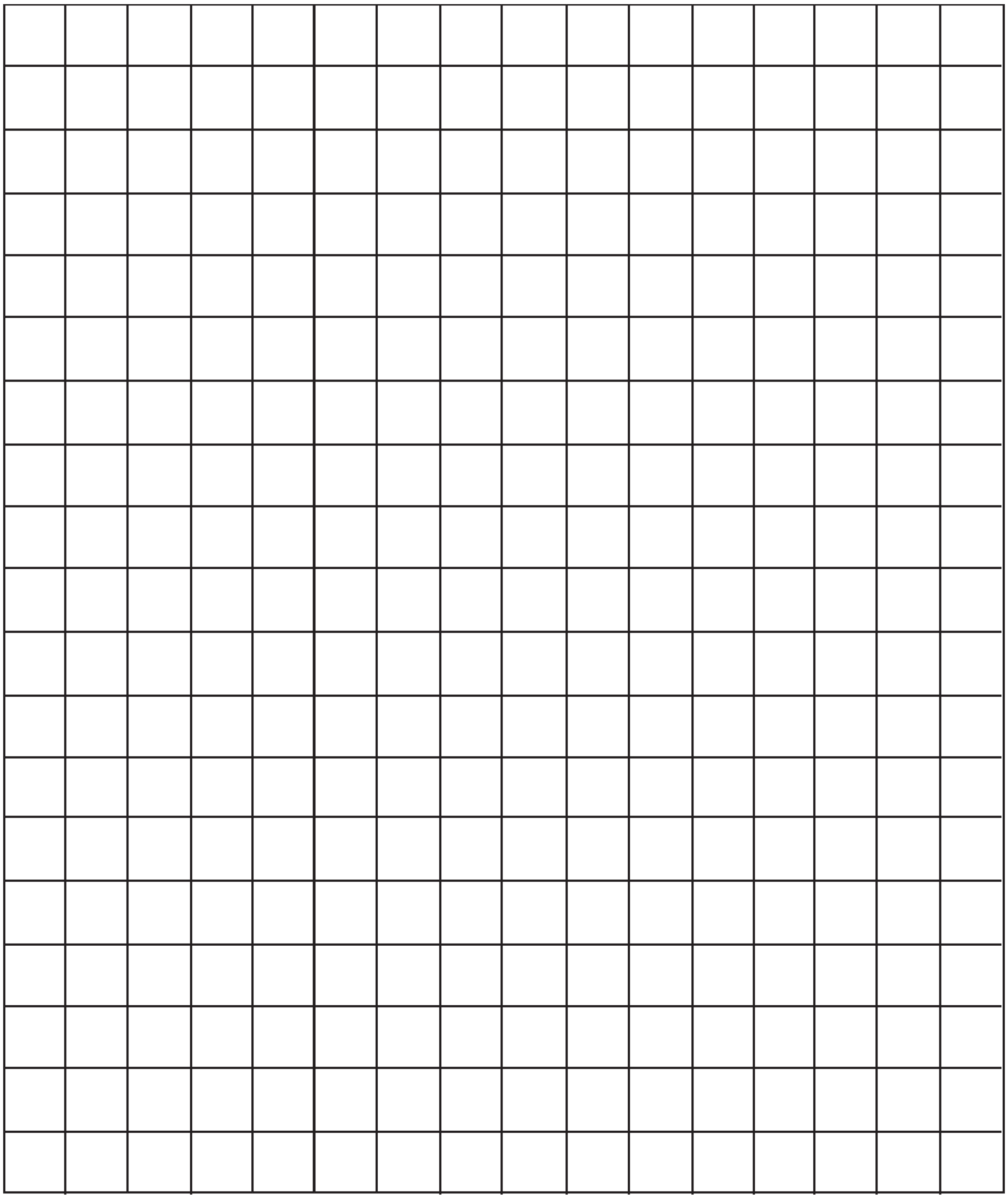
Quadrant: a division of the coordinate grid into patterns of ordered pairs



Naming Coordinates



Centimeter Grid Paper



Assignment #1: Probability and Integers

The following assignment is worth 130 points. The focus of this assignment is to practice probability skills and master integer operations. There are two parts to the assignment.

***Part 1*: (65 points)**
Use a traditional die (single for dice) for the following questions. Make sure to reduce fractions where appropriate.

1. List the sample space for a single die.

2. What is the probability of rolling a 3 on a single roll of the die?

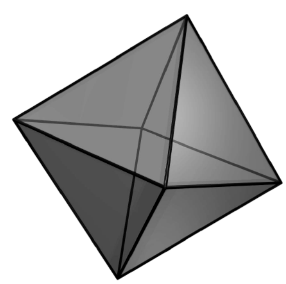
3. What is the probability of rolling an even number on a single roll of the die?

4. Explain your method of reasoning for #3 above.

5. What is the probability of *not* rolling a 4 *or* 5?

6. Explain your method of reasoning for #5 above.

7. Answer the following imagining an eight-sided die with numbers 1-8.
   - List the sample space for a single die.
   - What is the probability of rolling a 2 on a single roll of the die?
   - Are you more likely to roll a 2 on a 6-sided or 8-sided die? Explain your answer.



Part 2: **(65 points)**

Evaluate the following problems WITHOUT a calculator. Show enough work to support your answer.

1. $4+(-7)=$

2. $-3-(-7)=$

3. $(-9) \bullet (-6)=$

4. $35 \div (-5)=$

5. $(-2) + (-5) =$

6. $3 \bullet (-4) =$

7. $-1 - 4 =$

8. $-1 + 2 - 3 =$

9. $(-1) \bullet (-2) \bullet (4) =$

10. Draw a chip (or charged particle model) or number line to demonstrate the following problem. Explain your diagram. Include information on the resulting sign choice.

    $(-2) + 4$

Module B
Transparencies

Module B: Patterns and Predictions

The paraeducator will:

- **Employ strategies of problemsolving to make predictions and determine the probability of an event**
- **Develop integer concepts from concrete experiences**
- **Develop rules for integer addition and subtraction from concrete experiences**
- **Develop rules for integer multiplication and division from analyzing patterns**
- **Explore the coordinate graph system**

Probability Review

Probability:

- The chance of an event occurring
- Value between 0 and 1
- Requires the use of fractions
- Determined by the number of possible outcomes

Equally likely:
There is equal *probability (chance)* for an outcome.

Fair:
Object or event has options that are equally likely.

Likelihood:
How probable or likely is it that something will happen.

Likelihood is also known as **probability**.

Impossible: Event will never occur.

Certain: Event will absolutely occur.

Using Probability and Prediction

- Predict behaviors

- Explain past occurrences from patterns or data

- Real-life examples:
  - ▲ The gaming (casino) industry relies on theoretical probability to make a profit
  - ▲ Weather forecasters
  - ▲ Medical industry
  - ▲ Insurance companies

Naming Probability

For a coin, heads should show up 50% or 1/2 of the time.

$$P(H)=1/2$$

says the probability (P) of getting a heads (H) is 1/2

Determining probability:

1. Sample Space (S)

Set of all possible outcomes

- Coin: $S=\{H,\ T\}$
  S = sample space
  H = heads
  T = tails
  { } imply a set or grouping

- This becomes the *denominator* of the fraction to represent the probability

2. Event

Any subset or subgroup of the sample space

- The event becomes the numerator
- For the coin, heads (H) is one of two possible events

3. Determine the probability

a) Theoretical probability:

- A mathematical computation where A is the defined event; may be a single event or multiple events

$$\frac{\text{Number of elements in A}}{\text{Number of elements of sample space (S)}} = \frac{n(A)}{n(S)}$$

b) Experimental probability:

- Same calculation but the data come from an experiment

Making Heads or Tails of Probability

- ▲ What is the probability of getting a *heads or tails* on one flip?

- ▲ What is the probability of getting a *star* on a coin?

- ▲ What is the probability of getting a *heads and a tails* on one flip?

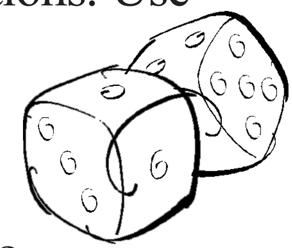
Mutually Exclusive

When event A occurs, event B cannot occur.

- For multiple events, probabilities for mutually exclusive events may be *added* together to determine the final probability

Give It a Chance

A bag contains 3 green golf tees, 2 blue tees, and 5 red tees. Use this information to answer the following questions. Use proper probability notation in your answers.



1. What is the probability of choosing a red tee?

2. What is the probability of choosing a blue tee?

3. What is the probability of choosing a green tee *and* a blue tee, $P(G \cap B)$, in one draw?

4. What is the probability of choosing a yellow tee?

5. What color tee is the *most* likely to be chosen on a single draw? Explain.

6. Are the outcomes of a single draw *mutually exclusive*? Explain.

7. What is the probability of drawing a red tee *or* a green tee, $P(R \cup G)$.

8. What is the probability of drawing a tee that is *not* green?

Integers

Integers

Positive and negative whole numbers.

Comparing Integers

Use > or < to compare the following values.

a) 6 _____ 8

b) 0 _____ -2

c) -2 _____ -4

d) -6 _____ 3

e) -5 _____ -7

f) -1/2 _____ -2

Integers

Adding Integers

To add like signs,

To add unlike signs,

Making zero

Adding a positive and negative (opposites) of the same number cancels to be zero.

Chip or charged particle model: 2 + (-3)

Integers

Subtracting Integers:

$-3 - (-4) \longrightarrow -3 - (-4)$

Steps:

1.

2.

$2 - (-5)$

$1 - 3$

$-4 - 4$

Integers

Multiplying Integers:

Link to addition: $2 + 2 + 2 + 2$

$(-3) + (-3) + (-3) + (-3)$

$(-5) + (-5) + (-5)$

$(-4) \times (-4)$

Generalization:

Positive product:

Negative product:

Integers

Dividing Integers:

Same rules as for multiplication of integers.

1. -10/2

2. -15/-5

3. 16/-4

4. 60/10

Integer Practice

Addition:

- Solid 2 and solid 3
- Outlined 1 and outlined 3
- Outlined 3 and outlined 4
- Solid 2 and outlined 3
- Outlined 4 and solid 5
- Outlined 1 and solid 3

Integer Practice

- Outlined 3, heads, outlined 4, tails
- Solid 2, heads, outlined 5, tails
- Solid 1, heads, solid 3, tails
- Outlined 4, heads, solid 4, tails

Operation Integer

Agent "Math"ew was assigned the duty of cracking the Integer Code. He only remembers his basic multiplication rules from elementary school. He must crack the code to save the world. The future of mathematics depends on it. Use patterns to complete each set.

1.

3	x	3	=	9
3	x	2	=	6
3	x	1	=	3
3	x	0	=	0
3	x	-1	=	-3
3	x	-2	=	-6
3	x	-3	=	-9
3	x	-4	=	-12

2.

4	x	4	=	16
3	x	4	=	12
2	x	4	=	8
1	x	4	=	4
0	x	4	=	0
-1	x	4	=	-4
-2	x	4	=	-8
-3	x	4	=	-12

Operation Integer

3. Write a rule for multiplying a positive and negative number.
When multiplying a positive and negative number, the product is always negative.

-2	x	2	=	-4
-2	x	1	=	-2
-2	x	0	=	0
-2	x	-1	=	2
-2	x	-2	=	4
-2	x	-3	=	6
-2	x	-4	=	8
-2	x	-5	=	10

4.

3	x	-5	=	-15
2	x	-5	=	-10
1	x	-5	=	-5
0	x	-5	=	0
-1	x	-5	=	5
-2	x	-5	=	10
-3	x	-5	=	15
-4	x	-5	=	20

5. Write a rule for multiplying two negative numbers.
When multiplying two negative numbers, the result is always positive.

Operation Integer

6. **$2 \times 2 \times 2 \times 2 = 16$**
   $2 \times 2 \times 2 \times (-2) = -16$
   $2 \times 2 \times (-2) \times (-2) = 16$
   $2 \times (-2) \times (-2) \times (-2) = -16$
   $(-2) \times (-2) \times (-2) \times (-2) = 16$

7. **Write a generalization that you see from #6.**
   The number of negatives in a problem affects the final product. An even number of negatives makes a positive product. An odd number of negatives makes a negative product.

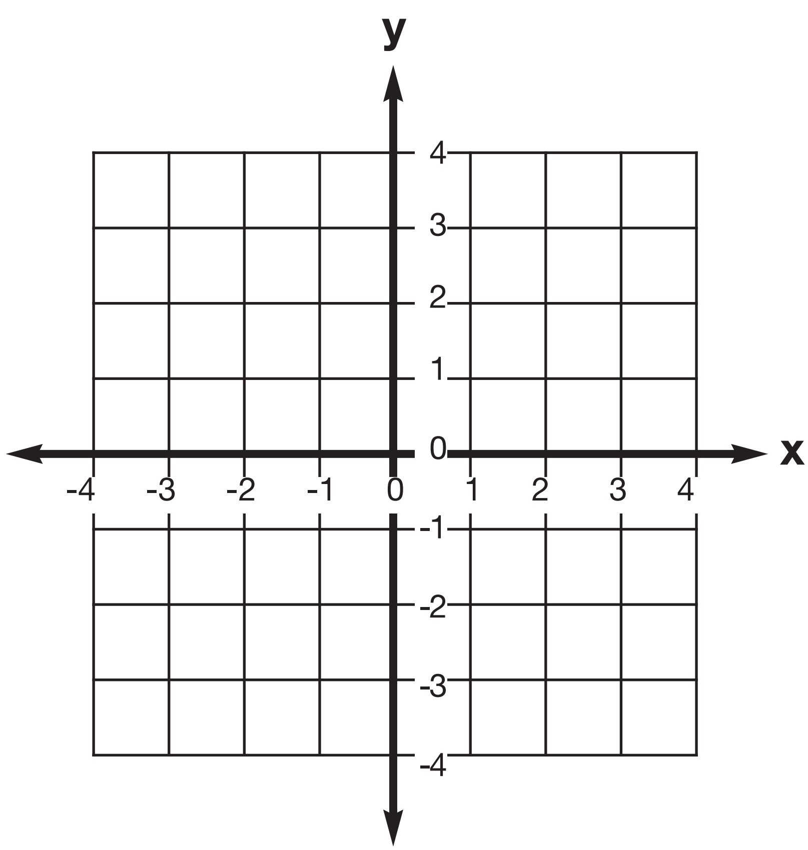
Coordinate Graphing

Cartesian coordinates or coordinate graphing:

Requires ordered pairs and axes.

***x axis*: horizontal axis**

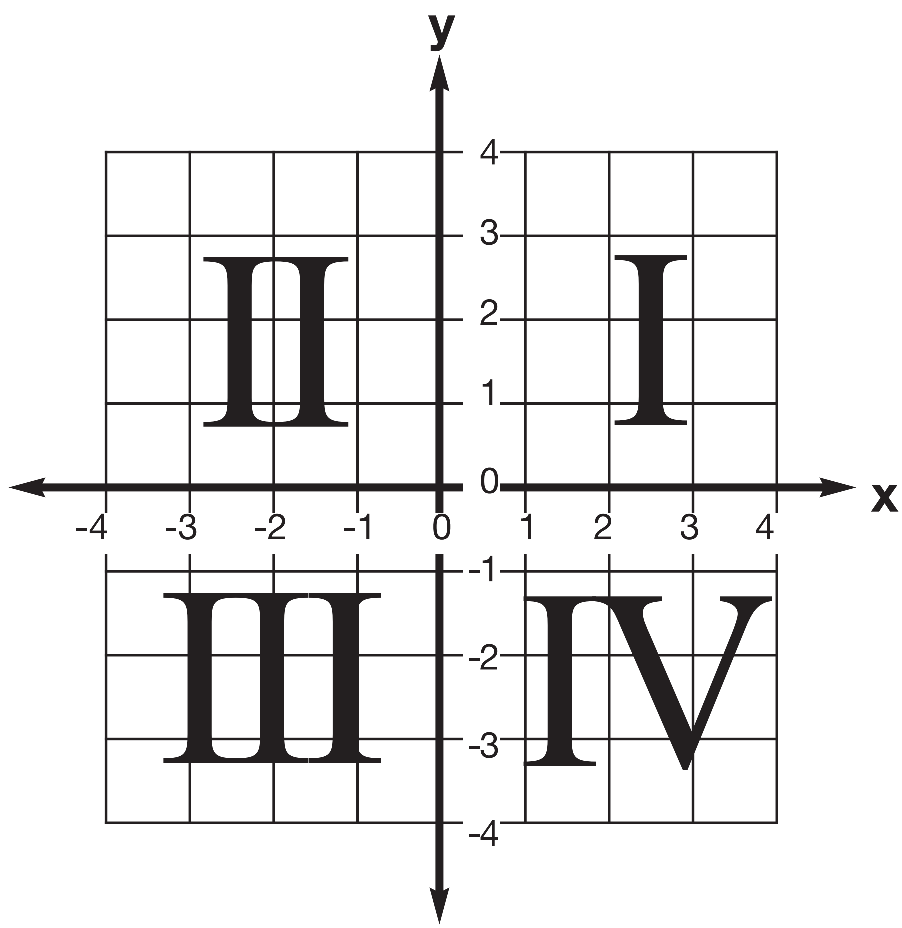
***y axis*: vertical axis**



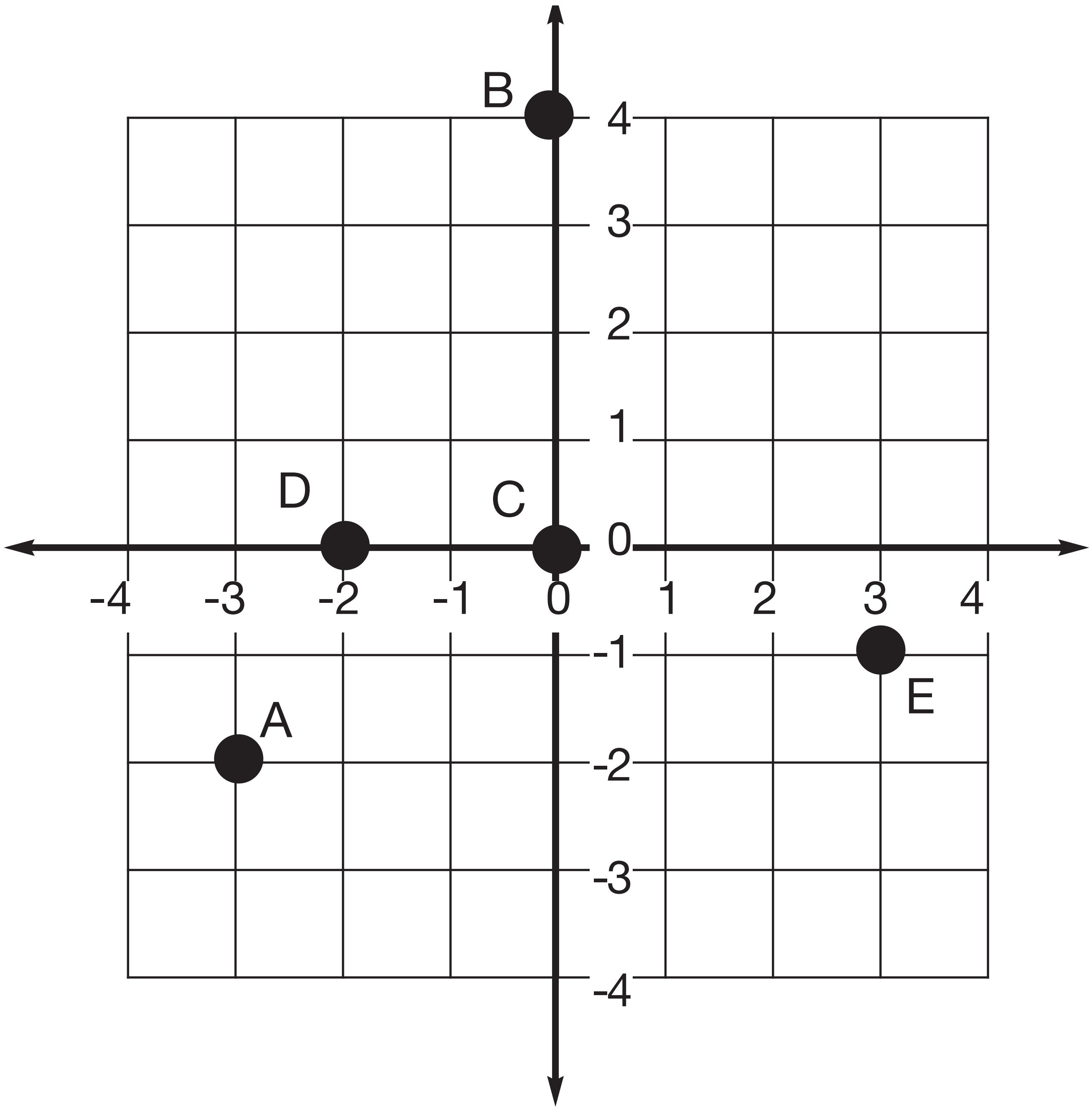
***Coordinate or ordered pair*:**
location in the form (x, y)

Coordinate Graphing

***Quadrant*:**
a division of the coordinate grid into patterns of ordered pairs



Naming Coordinates



Module C
Instructor's Guide

Module C: Algebraic Fundamentals

Module Goals
Use the transparency **Module C Goals (T1)** to review the module goals.

Module C: Algebraic Fundamentals
The Paraeducator will:
1. Use patterns and sequences to predict and generalize outcomes
2. Describe patterns and other relationships using words and expressions
3. Relate basic patterns to algebraic concept development
4. Develop a plan for solving basic algebraic equations

Patterns have been the focus of the two prior Academies; they are also a major focus here. The ability to generalize a pattern to make a prediction is the key to algebra. Many students fear algebra because of the use of letters. Students need concrete experiences with prediction before they can see the purpose of algebraic concepts. With background experience, students begin to naturally look for patterns and come to view writing and solving algebraic expressions and equations as one more tool to make problem solving more efficient.

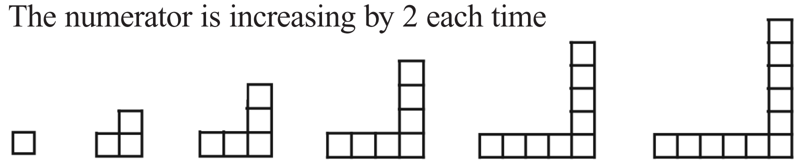
Goal 1: Use patterns and sequences to predict and generalize outcomes.

1.1 Activity: What's Next?

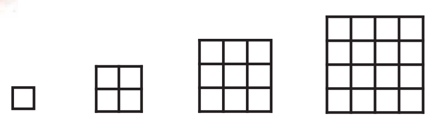
The paraeducator will use prediction skills to complete patterns and generalizations.

Materials:
- Handout **What's Next? (H1)**

1.1.1 Steps
- Individually or in pairs, have participants complete the handout **What's Next?**
  - ▲ Many participants will find this challenging and demonstrate some frustration
  - ▲ It is important that participants work through each pattern as best they can. This will get them started on the processes for generalizing patterns that are required for algebra
- Discuss the results as a group.
  - ▲ Be flexible with regard to the pattern descriptions, as that topic will be addressed later in this module
- *Answers:*
  1. 5, 9, 13, 17, 21, <u>25</u> , <u>29</u> , <u>33</u>, <u>37</u>
     - o This pattern adds 4 each time
  2. 13, 10, 7, 4, 1, <u>-2</u>, <u>-5</u>, <u>-8</u>
     - o This pattern subtracts 3 each time and goes into the negative numbers as it passes zero

3. 1, 3, 7, 13, 21, 31, 43, 57, 73
    - This pattern increases with a pattern of +2, +4, +6, etc.
4. 7, 14, 21, 28, 35, 42, 49, 56
    - This pattern may be listed in two different ways:
        - Adding 7 each time
        - Multiplying 7 by increasing factors (x2, x3, x4, etc.) from 7
5. 128, 64, 32, 16, 8, 4, 2, 1, 1/2
    - This pattern requires going in both directions
    - From the given data, it appears that each number is cut in half
    - To get the first number, the number must be doubled
6. 4/5, 6/5, 8/5, 2, 12/5, 14/5, 16/5, 18/5, 4
    - Fractions often make patterns difficult to see. This pattern will be the most challenging for participants
        - Participants will have difficulty with the 2 and 4
        - They may need assistance in rewriting those fractions asn10/5 and 20/5, which equal 2 and 4, respectively.
    - It should be obvious that the denominator does not change
    - The numerator is increasing by 2 each time
7. 
    - Several approaches may have been used to complete this pattern:
        - Total number of blocks (odd numbers)
        - A block added to each leg
        - Number of blocks in each leg is the position in the sequence (For example, the second picture has 2 squares in each leg and is second in the sequence; the third picture has 3 squares in each leg and is third in the sequence)

    ***Note to Instructor**: It is likely that no one will get this pattern. Do not mention if it does not occur naturally.

8. 
    - The next two patterns will have 25 (5x5) and 36 (6x6), respectively

    **** Note to Instructor:*** Some participants may recognize these numbers as perfect squares from prior Academies. Do not mention it if it does not occur naturally.

1.2 Lecture: Types of Patterns

Note to Instructor: Remind participants to use their Math Journals to take notes during lecture periods.

The Grades K-4 Academy starts this process with concrete materials. The activity "What's Next?" continues the basic process from grades K-4 but is more abstract and depends on an understanding of numbers and operations. This activity is used as an introduction to algebra because algebra is about prediction and generalization. The ability to predict depends on the ability to recognize a pattern.

In the Grades K-4 Academy, two types of sequences or patterns were introduced. Use the transparency/handout **Types of Patterns (T2/H2).**

> ***Repeating patterns:*** Elements in the patterns repeat according to some rule. Repeating patterns are an easy place to begin helping students with prediction skills.
>
> ***Growing patterns:*** Elements in the patterns change (grow) according to some rule. These types of patterns are the introduction to algebraic concepts. The simplest growing pattern is counting.

The concept of *growing patterns* expands in grades 5-8 with two types of growing patterns, arithmetic sequence and geometric sequence.

> ***Arithmetic sequence:*** A sequence where each successive term is obtained from the previous term by the addition or subtraction of a fixed number, the **difference**.
>
> Example: 1, 4, 7, 10, … The constant or **difference** here is 3.
>
> ***Geometric sequence***: A sequence where each successive term is obtained from its predecessor by multiplying or dividing by a fixed number, the **ratio.**
>
> Example: 2, 4, 8, 16, 32, … The **ratio** here is 2.

From the handout **What's Next? (H1)** participants should be able to recognize the above types of patterns.

> *Arithmetic:* #1, 2, 4, 6, 7
> *Geometric:* #5
> *Neither:* #3, 8

Recognition of pattern types helps students verbalize what is happening and eases the move to a general algebraic expression.

Goal 2: Describe patterns and other relationships using words and expressions.

2.1 Discussion: Using Words to Describe Relationships

The paraeducator will use words to describe patterns or relationships.

Materials:
- Completed handout **What's Next? (H1)**
- Transparency **Math Words (T3)**

2.1.1 Steps
- Go back to the handout **What's Next? (H1)** and focus on the descriptions of patterns that participants saw.
- Use the transparency **Math Words (T3)** to record a class list of math words used to describe various patterns. Some open discussion may help participants rephrase their thinking.

  Math Words: (possible responses)
  ▲ Added (as in added 2 each time)
  ▲ More (as in 3 more)
  ▲ Less (as in getting smaller)
  ▲ Subtracted
  ▲ Twice or doubled
  ▲ Multiplied
  ▲ Divided
  ▲ Squared

- It is important to identify common math words as these words have mathematical translations. Mathematical translations are numerical expressions.
- Review the terms *algebra* and *variable* from the transparency/handout **Keys to Algebra (T4/H3).**

  Algebra:
  Algebra is a method by which to describe and predict the behavior of aset of data. Data can be anything: blocks, toys, weather, numbers, etc.

  A Variable:
  ▲ Symbol, object, or letter that can change value depending on the problem in which it appears
  ▲ Simply holds a place until is has a value, then it changes with the next problem
  ▲ Has a value that can vary, or is "variable"

- In order to make sense of algebra, it is necessary to use our language skills to give meaning to abstract mathematical concepts. A key skill in algebra is to translate descriptions into math terms and mathematical sentences into regular language. This solidifies the concept that algebra is just shorthand for everyday experiences.
- Introduce and discuss the words listed for various operations using the transparency/handout **Keys to Algebra (T4/H3).**

Addition (+)	**Subtraction** (-)	**Multiplication** (X)	**Division** (÷)
Add	Subtract	Multiply	Divide
Added to	Subtracted from	Multiplied by	Divided by
Sum	Difference	Product	Quotient
Total	Minus	Times	
Plus	Less than	Of	
More than	Decreased by		
Increased by	Take away		

- Start with simple group practice to go from a phrase to an expression. Record the mathematical expressions on the transparency/handout **Keys to Algebra (T4/H3).**

Note to Instructor: It may be helpful to substitute real numbers for the variables until the participants get used to writing expressions. For the first example, ask what "4 more than 5 equals?, 4 more than 6 equals?" and follow with the general expression by asking for the process from the concrete examples, which added 4 each time to the given number.

▲ 4 more than *p* (4 + p or p + 4)

▲ 5 less than *m* (m – 5)
  - Order matters in subtraction

▲ y decreased by 10 (y – 10)

▲ the quotient of a number and 7 (*x*/7)
  - If order is not specifically stated, list the quotient in the order given
  - Order matters for division

▲ the product of 6 and an unknown (6*h*)
  - Note that in multiplication, no symbol is necessary between the number and the variable

Note to Instructor: It is important that participants realize that the unknown or number may be represented by a variable of any letter. It is too easy to overuse x. Make sure to vary letters.

- Practice going the other direction: from expressions to words. This direction is commonly more difficult because the wording can be confusing. Record participants' responses on the transparency/handout **Keys to Algebra (T4/H3).** The following are suggested responses.
  - ▲ n + 4 (4 more than n or an unknown)
  - ▲ 6 – z (6 less z; z less than 6)
  - ▲ 10/*y* (10 divided by y; the quotient of 10 and y)
  - ▲ 8*b* (8 times an unknown or b; the product of 8 and b)
  - ▲ 5*x* + 1 (1 more than the product of 5 and a number)

Note to Instructor: Make sure to note that unknowns or variables may be any concept, such as the number of legs, coins, ages, etc.

2.2 Activity: Expressing Expressions

The paraeducator will translate and write mathematical expressions to represent real life situations.

Materials:
- Transparency/handout **Expressing Expressions (T5/H4)**

2.2.1 Steps
- Explain that the expressions done during the prior discussion were general phrases. Without defining the meaning of the variable, the expressions had little purpose.
- Expressions are necessary for generalizing multiple problems into a single expression that can be reused as the data change.
- Go over the first example. Ask participants to think how they would act out the situation if the numbers were known to determine what operation is necessary.
  - ▲ Amount of money in dimes
    - o Determine a variable and define it
      - d = the number of dimes
    - o Determine the required operation with examples
      - 5 dimes would be $0.50
      - 3 dimes would be $0.30
      - 7 dimes would be $0.70
    - o Each time the number of dimes was multiplied by $0.10, which is the amount of the dime
    - o The expression would be 0.10 x *d* or 0.10*d*, which means that for every number of dimes, the amount would be multiplied by 0.10
- In pairs or small groups, have participants complete the handout **Expressing Expressions (T5/H4)**.

- Use the transparency **Expressing Expressions (H4)** to go over the activity.
  1. Amy's age in 5 years:
     A = Amy's age now
     A + 5 Amy's age in 5 years

  2. The number of hours remaining from a 12-hour driving trip:
     h = # of hours driven
     12 – h Numbers of hours remaining

  3. The number of cookies for each person if 50 cookies were shared equally:
     C = # of people at the party
     $50/c$ Number of cookies for each person

  4. The number of game cards for Matt and Rosie combined if Rosie has 20 cards:
     m = # of cards Matt has
     $20 + m$ Number of cards they have total

  5. The number of pictures in Juan's scrapbook if each page holds 10 pictures:
     p = # of pages in the scrapbook
     $10p$ Number of pictures in the book

  6. Let m equal the number of motorcycles in a parking lot and c equal the number of cars in a parking lot. Write an expression for each of the following:
     a. The total number of wheels on all the motorcycles in the parking lot
        $2m$ Each motorcycle has 2 wheels

     b. The total number of wheels on all the cars in the parking lot
        $4c$ Each car has 4 wheels

     c. The total number of wheels in the parking lot on all the cars and motorcycles
        $2m + 4c$ Total for all wheels

Goal 3: Relate basic patterns to algebraic concept development.

3.1 Discussion: Evaluating Expressions

The paraeducator will evaluate algebraic expressions to analyze relationships.

Materials:

- Handout **Expressing Expressions (H4)**
- Transparency/handout **Keys to Algebra (T4/H3)**

3.1.1 Steps

- Up to this point, expressions have been used to represent abstract phrases and concrete examples, but no attention has been paid to evaluating or giving numerical meaning to the expressions.
- To be successful, middle school students need to understand that expressions are simply "rules" for how given data will be manipulated.
- It is useful to create a table to evaluate an expression. Evaluating expressions may be thought of as inputs and outputs.
  - ▲ Return to the first example on the **Expressing Expressions (H4)** handout

    Amount of money in dimes

    d = the number of dimes

    Expression: 0.10 x *d* or 0.10*d*
  - ▲ The expression acts as a rule that says "for each number of dimes, multiply by 0.10, the value of the dime"
    - o Another name for the "rule" is a *function* (see the transparency/ handout **Keys to Algebra [T4/H3]**)
    - o Explain that a *function* is like a machine: It takes in inputs and "spits" out outputs

***Note to Instructor*:** The term *function* has a rather complex definition mathematically but for the purpose of this lesson, it will be treated as a "rule" to make participants comfortable with the term.

- ▲ After choosing values for *d*, we *substitute* or replace *d* with our values and follow the rule (see transparency/handout **Keys to Algebra [T4/H3]**); the chosen values are the inputs

Substituting:

Replace the variable with a numerical value

- o It might be helpful to use if-then statements (these may be thought of as conditions for the rule):

  if $d=0$, then

  $0.10d$

  $0.10(0) = 0$

  if $d=1$, then

  $0.10d$

  $0.10(1) = \$0.10$

d	*0.10d*	=
0	0.10 (0)	0
1	0.10 (1)	$0.10
2	0.10 (2)	$0.20
3	0.10 (3)	$0.30
4	0.10 (4)	$0.40
5	0.10 (5)	$0.50
6	0.10 (6)	$0.60

This table confirms our expression as each product or *output* increased by one dime

- Ask participants if zero was an appropriate value to substitute. (yes, because you can have zero dimes)
- Ask participants if negative numbers are appropriate to substitute in this expression. (no, it is not possible to have a negative number of dimes)
- Ask participants for scenarios where negative numbers would be appropriate. (examples: temperature, depth, money if debt were a consideration)
- In pairs, have participants choose two of the problems from the handout **Expressing Expressions (H4)** and create an input/output (in later grades this becomes domain and range) table for the general expression or function.

***Note to Instructor**:* Check for accuracy of group work in setting up the table and evaluating the expressions. This will be on the homework practice.

- ▲ Use at least four inputs for each chosen expression
- ▲ Ask participants to check for the following:
  - o Is the variable appropriate to produce the requested answer? (on #5 it is easy to mistake the variable pictures for pages to produce the total number of pictures)
  - o Are negative values appropriate for evaluation?
  - o Does the expression produce expected results?

- The expressions listed on the handout **Expressing Expressions (H4)** were primarily single-step expressions. In true mathematics, few expressions or problems consist of a single step. Remind participants that in the case of multiple steps they are to follow the order of operations. Show these examples on the transparency/handout **Keys to Algebra (T4/H3).**

***Note to Instructor:** Make sure that when participants substitute a value, the substitution is put in parentheses. The reason for this practice is that it makes it easier to find the value and to check for errors.*

- ▲ If $p = 0$, then $2p + 3 = ?$ (3)
  $2p + 3$
  - o $2(0) + 3$
    $0 + 3 = 3$
  - o For this example, remind participants of the order of operations, which says that multiplication/division comes before addition/subtraction
- ▲ If $n = -2$, then $-4n - 1 = ?$ (7)
  $-4n - 1$
  - o $-4(-2) - 1$
    $8 - 1 = 7$
  - o Go over this one carefully as a sign review. Reiterate the order of operations
- ▲ If $n = 1$, then $(16 \div 8) - n = ?$ (1)
  $(16 \div 8) - n$
  - o $(16 \div 8) - 1$
    $2 - 1 = 1$
  - o This has multiple steps and requires the order of operations

- ▲ If $a = 2$, then $a^2 = ?$ (4)
    - a^2
    - o $(2)^2 = 4$
    - o This example is for a quick review of exponents
- It is important to close the discussion by explaining that it does not matter what values are input (within the appropriate range of the scenario) into the expression.
    - ▲ In the coin example, any value greater than or equal to zero may have been used
    - ▲ It is often a good idea to get a variety of samples such as what happens with $d = 10$ (the amount of money becomes $1 as participants know from life experience)

3.2 Activity: Guess My Rule

The paraeducator will use generalizations to create a rule for given data.

Materials:
- Transparency/handout **Guess My Rule (T6/H5)**
- Blank transparency

3.2.1 Steps
- When a scenario is not given, it is often necessary to use data, both inputs and outputs, and work backward to create a function.
- Use the following functions (or any of your choice) for the oral exercises listed below. This should not take much time as these are basic functions. Remind participants that the variable stands for the input.

    $x - 4$
    x
    $x + 4$
    $2x$
    $2x + 1$ (This is a challenge and may not be appropriate for all groups)

    - ▲ In their journals, have participants create a small input/output table for each oral exercise
    - ▲ For each exercise above, have participants make guesses to the machine
    - ▲ Evaluate the given input and produce the appropriate output on your own chart
    - ▲ Have participants record the data (both input/output)
    - ▲ Once a participant sees the pattern, have that participant suggest a rule (function) for the group in algebraic terms
    - ▲ Test the suggestion before revealing the actual function

***Note to Instructor**: It may be necessary to remind participants that they are looking at the relationship of what went in and what came out. They are not necessarily looking at the sequence produced as in the first activity of this module.

- Have participants work in groups of four to complete the handout **Guess My Rule (H5).**
- Check as a group by filling in the transparency **Guess My Rule (T6)**.

▲ Rule: q – 10

Input (q)	26	**39**	**55**	66	73	74	**93**
Output	**16**	29	45	**56**	**63**	**64**	83

▲ Rule: ___**3q**___________________________

Input (q)	9	17	25	33	41
Output	27	51	75	99	123

▲ Rule: ___**r + 11**___________________________

Input (r)	19	30	**41**	52	58
Output	30	**41**	52	63	69

▲ Rule: ___ $f \div 5$ or $f/5$ ___________________________

Input (f)	15	20	25	**30**	35	40	45
Output	3	**4**	5	**6**	7	8	**9**

3.3 Lecture: Patterns Producing Expressions

****Note to Instructor:*** Remind participants to use their math journals to take notes during lecture periods.

A unique aspect of algebra is that it is circular; the processes link one into the other and can be done in forward or reverse order, depending on the mathematical situation. As stated earlier, algebra is used to generalize and predict data. In its simplest form, input/output tables are a way to predict outcomes. These tables assume that the “rule” is already given or that something is known about the input/output values.

The most challenging skill is to predict outcomes when “rules” or expressions are not immediately given, and when only the outputs are known. Success with this skill is based on organizing data and being able to recognize patterns in the data.

Starting with a basic pattern such as 2, 4, 6, 8, …, most participants find it easy to see that this is counting by 2’s and can complete the next several terms. If asked to find the 100^{th} term in this pattern, the only choice would be to write them all out. This is time consuming and tedious. An algebraic expression may be used to create a general rule, whereby we could instantly calculate the 100th term.

The initial reaction when asked to create an expression is x + 2, because the numbers appear to be two apart or adding 2 each time. While this would work to complete a table of even numbers, it does not help us predict the 100^{th} term in the sequence, however. Our expression needs to be able to have an input of 100 which is the term number.

For the given sequence, 2 is the first term, 4 is the second term, 6 is the third term, and so on. This is easily organized into a table. Sometimes tables are the best way to see a connection between a sequence and its term number. Use the transparency/handout **Algebraic Patterns (T7/H6).**

Term number	1	2	3	4	5	6	7	*n*	100
Terms	2	4	6	8	10	12	14		

By looking at the term number and the term of the sequence, an obvious relationship of doubling or multiplying by 2 should appear. The concept of 2 is still present from the initial guess of x + 2, but the 2 comes with multiplication.

To create a general expression to find any term in the sequence, the *nth term* must be found.

> ***nth term***
> Asks for a general expression to predict any term of the sequence.

Each time the term number is multiplied by 2; therefore, *n* must be multiplied by 2. This gives a general expression or *nth term* as 2*n*.

Term number	1	2	3	4	5	6	7	*n*	100
Terms	2	4	6	8	10	12	14	2*n*	

To find the 100th term, we can substitute 100 for n, since n represents the term number.
So the 100th term is 2 x 100 = 200.

Term number	1	2	3	4	5	6	7	n	100
Terms	2	4	6	8	10	12	14	$2n$	200

Some patterns are more complex to see, and we must use what we already know from prior sequences. Consider the sequence of 1, 3, 5, 7, …

Term number	1	2	3	4	5	6	7	n	100
Terms	3	5	7	9	11	13	15		

It would appear that a similar general expression may be used with the odd number sequence as with the even number sequence above. If the term number is plugged into the expression $2n$, only even numbers are produced.

Term number	1	2	3	4	5	6	7
$2n$	2	4	6	8	10	12	14
Desired Terms	3	5	7	9	11	13	15

While the pattern is very close, it is not the sequence that is sought. If the data are analyzed carefully, the table shows that the $2n$ sequence put the sequence within 1 of the desired sequence. Adding 1 to each term will produce the desired sequence.
This gives a general expression or *nth term* as $2n + 1$.
The 100th term is then 201.

Term number	1	2	3	4	5	6	7	n	100
Terms	2	4	6	8	10	12	14	$2n+1$	201

This fits the odd-number expectation, and is close in result to the even-number sequence. We can assume that our result is correct.

A hint for looking at sequences such as the one above is to look at the difference between the sequence numbers. That difference will play out as the product with the term number. To reach the desired sequence term, adding or subtracting is necessary, making the rule a two-step process.

As patterns become more complex and less obvious, formulas are required. These will not be addressed here. Students who understand the basics of data organization, generalizations, and prediction can do more complex patterns with the formulas.

3.4 Activity: Guess My Rule II

The paraeducator will use generalizations to create a rule for a sequence using the term number.

Materials:
- Handout/transparency **Guess My Rule II (T8/H7)**

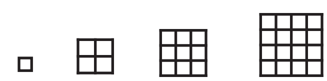
3.4.1 Steps
- In groups, have participants complete handout **Guess My Rule II (H7).**
  - ▲ The first two questions are from handout **What's Next? (H1)**
- Go over answers as a group by filling in the transparency **Guess My Rule II (T8).**

1. *nth term:* __________***7n***______________________

Term Number (n)	1	2	3	4	5	6	7	...100
Sequence	7	14	21	28	35	42	49	700

   - o This should be fairly straightforward
   - o Possible errors include n + 7 for participants who are focusing only on the sequence

2. *nth term:* ________ n^2 ____________________



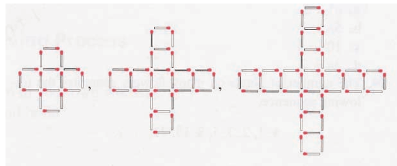
Term Number (n)	1	2	3	4	5	6	7	...100
Sequence	1	4	9	16	25	36	49	10,000

This pattern is a common sequence of perfect squares; participants may need a reminder of squares

3. *nth term:* ___________ $4n + 1$ __________________

Term Number (n)	1	2	3	4	5	6	7	...100
Sequence	5	9	13	**17**	**21**	**25**	**29**	**401**

Matchsticks:



- o This pattern is the most difficult, as there is no single-step rule to get from the term number to the sequence term
- o Remind participants to look at the sequence to find the difference; that becomes the product

  The difference is 4 in the sequence:
  4(1) = 4
  4(2) = 8

  This is close to the sequence but not completely correct. If 1 is added to each, that will be the sequence.

Note to Instructor: All these sequences are arithmetic in nature. Geometric sequences are more difficult to do without formulas and are not addressed here.

Goal 4: Develop a plan for solving basic algebraic equations.

4.1 Discussion: Solving Basics

The paraeducator will develop basic solving skills from inspection (trial) processes.

Materials:

- Transparency/handout **Solving Basics (T9/H8)**

4.1.1 Steps

- Point out that a major skill in algebra is solving equations (see the transparency/handout **Solving Basics [T9/H8]**).

  Equation:
  An expression (sentence) with an equals sign.

  Solution:
  A number that makes the equation true.

- Explain that *equations* differ from expressions in two major ways:
  - ▲ Contain an equals sign
  - ▲ The goal is to find all the values for the variable that make the statement true
- Explain that participants and students have been doing algebra since early elementary school when they were asked for missing numbers such as

  3 + ✸ = 7
  What is sunshine?

- Note the key word "is" in the above statement. The word "is" signals that this is a statement, similar to a regular sentence with a verb. This mathematical sentence is an *equation*.
- Have participants discuss possible solutions via inspection (or trial) for each equation (use the transparency/handout **Solving Basics [T9/H8]**).
  - ▲ Read each example aloud using the word "is" at the equal sign to show that it is an *equation*
  - ▲ For each problem, ask participants to discuss what they are thinking in solving these algebraic equations by inspection
    - o $x + 9 = 17$(x = 8)
    - o $3y = 12$ (y = 4)
    - o $18 = m - 7$ (m = 25)
    - o $z \div 4 = 4$ (z = 16)
  - ▲ Once completed, ask if there could be more than one solution for each equation listed here. (no, there is only one answer to each of these problems)

***Note to Instructor**: These equations are linear equations so there is only one solution to each equation. This will be addressed in the following module.

- Some participants may be doing the problems by trial and error. Others may have progressed to thinking about the opposite processes to get the answer.

4.2 Activity: Maze Mischief *(optional)*

The paraeducator will use basic games to understand the solving concept for algebraic equations.

Materials:
- Handout **Maze Mischief (H9)**

4.2.1 Steps
- Give each individual the handout **Maze Mischief (H9).**
- Assign half of the group to complete the maze from "Start" to "Finish."
- Give each person who completed a maze a partner with a blank maze.
- Tell participants who have a blank maze to turn their maze upside down so that "Start" is at the bottom.
- Ask the person who completed the maze to explain to his or her partner how to solve the maze from "Start" to "Finish" without showing the solution,
  - ▲ The person following directions may not ask any questions
  - ▲ The person giving directions may not touch the partner's maze
  - ▲ The person giving directions is not supposed to talk while the other person is drawing
- As a group, discuss what made this difficult.
  - ▲ The person who initially completed the maze has a certain path in mind
  - ▲ The person giving directions has to reverse the thinking involved
  - ▲ Physical orientation of the maze makes judging distance and the timing of turns difficult

4.3 Lecture: Patterns Producing Expressions

Note to Instructor: Remind participants to use their Math Journals to take notes during lecture periods.

The purpose of the maze activity was to get participants to think in reverse. When solving algebraic equations, everything must work in reverse or opposites to get back to the values for the variable that makes the equation true. The ultimate goal is to *isolate the variable.*

The previous example, $x + 9 = 17$, could be solved by trial and error. However, as problems get more complex, trial and error is not possible. There must be a plan for solving all types of equations.

For middle school students, solving problems first with inspection sets the stage for creating solving rules. A common error is to rush to the rules before a student understands the need for the rules.

The problem $x + 9 = 17$ has a solution of x = 8. It is important to discuss methods for getting 8 with the information given. The point should arise that by subtracting 9 from 17, we get the number 8. It would appear that using the opposite operation of subtraction to solve the addition problem would provide the correct answer.

The problem $18 = m - 7$ has a solution of $m = 25$. This solution was obtained by adding 18 and 7 together.

The problem $3y = 12$ has a solution of $y = 4$. This solution was obtained by dividing 3 into 12. This finds the missing factor. The answer makes sense, because the factors are smaller than the final product.

Finally, the problem z ÷ 4 = 4 has a solution of $z = 16$. We know this makes sense as the initial number must be larger than 4 to get a whole number.

From these examples, the concept of opposites should become apparent. Reversing the current operation is the key to solving the equation.

An important aspect of solving algebraic equations involves the concept of equality. An algebraic equation should be thought of as a balance. The equals sign says that the two sides of the equation should be equivalent to each other or "balanced." To make this happen, use the phrase "whatever you do to one side, do to the other." Organization of work in algebra is the key to successfully solving more complex problems.

Using the same examples, show the appropriate work with the problems on the transparency/handout **Solving Basics (T9/H8)**. Note that work can be done vertically or horizontally; it is a learner preference. When working horizontally, it is important to point out that order matters in the placement of the values. Both methods with be shown for each example.

Some general suggestions:

- Show work on both sides
- Cross out concepts that cancel
- Keep equals signs aligned (in vertical method)

It is helpful to engage participants in dialogue such as asking the following:

- What did they do to the variable? ("they" is just a way of asking about the setup)
- I will do the opposite, which is …

Example 1:

- What did they do to the variable?
- They added 9
- I will subtract 9

***Note to Instructor**: Suggest to paraeducators that it is helpful to write the problem in one color and show the work in another color so that their students can follow the process.

In addition and subtraction, it is important to discuss what happens to the number on the side with the variable. For the first problem, it is important to note what happens to the +9 on the left side of the equation. It is *canceled to zero* conceptually in one of two ways: +9 and -9 canceling, or subtracting 9 as the opposite of adding 9. This discussion is quite complex later in algebra because students read these concepts differently. No one method is more correct than another, but differing interpretations can cause large gaps in students' understanding.

$$\begin{array}{lcl} x + \cancel{9} = 17 & & x + 9 = 17 \\ \underline{\quad \cancel{-9} \quad -9} & \text{or} & \cancel{x + 9 - 9} = 17 - 9 \\ x \qquad = 8 & & x = 8 \end{array}$$

$$\begin{array}{lcl} 18 = m - \cancel{7} & & 18 = m - 7 \\ \underline{+7} \qquad \underline{\cancel{+7}} & \text{or} & 18 + 7 = m - 7 + 7 \\ 25 = m & & 25 = m \end{array}$$

For every problem solved, it is important to substitute the answer back into the equation to check for accuracy. This is one positive aspect of algebra; the work can be checked for accuracy. Note that the two sides of the equations are kept as separate problems; checking verifies they are truly equal.

$(8) + 9 = 17$?	$18 = 25 - 7$?
$17 \quad = \quad 17$	$18 = 18$
✔ (this checks)	✔ (this checks)

The notation for multiplication and division varies somewhat by teacher preference. Sometimes, using the traditional ÷ symbol introduces errors as students are unsure where to put it. It is a mathematical preference to treat division as fractions. Both will be shown here.

It is also important to discuss what happens to the numbers with the variables in multiplication and division. They do not cancel to zero as in addition and subtraction. When we multiply or divide (as in the opposite process) by the same number, the numbers *cancel to 1*. That means there is a "hidden" 1 with the y in the following problem. In algebra, our goal is to isolate the variable to a single variable so 1y is the same as y.

$$\frac{\cancel{3}y}{\cancel{3}} = \frac{12}{3}$$
$$y = 4$$

or

$$3y = 12$$
$$\cancel{(3 \div 3)}y = 12 \div 3$$
$$y = 4$$

$$\cancel{4}\left(\frac{z}{\cancel{4}}\right) = 4 \cdot 4$$
$$z = 16$$

or

$$z \div 4 = 4$$
$$\cancel{z\ (\div\ 4 \cdot 4)} = 4 \cdot 4$$
$$z = 16$$

Checking:

3(4) = 12 ?
12 = 12
✔ (this checks)

(16)/4 = 4?
4 = 4
✔ (this checks)

It is important to reiterate that whatever is with the variable, including the sign, must go away. The equation should solve down to a positive variable to ease the checking process. See the example below.

$$\frac{\cancel{-2}y}{\cancel{-2}} = \frac{18}{-2}$$

Remember that two negatives make a positive.

$$y = -9$$

Checking:

-2 (-9) = 18 ?
18 = 18
✔ (this checks)

While not a key skill for this module, some attention should be paid to simple multi-step equations such as $2n + 1 = 7$ (see transparency/handout **Solving Basics [T9/H8]**). Talk through these as a group. Point out to participants that there are two separate issues: one multiplication and one addition. The concept of opposites still applies. However, the order of operations is in reverse.

Review order of operations:

PEMDAS (Please Excuse My Dear Aunt Sally)

Parentheses (or other enclosures)
Exponents
Multiplication/division (as they occur from left to right)
Addition/subtraction (as they occur from left to right)

Reversing the order of operations means that addition/subtraction operations must occur before any multiplication/division operations. In general, multiplication/division should be the very last step in an equation and should be done only once. A good analogy is a knot where the variable is at the center. The pieces furthest from it must go first or untangle; the goal is again isolating the variable.

Example:

$$\begin{aligned} 2n \cancel{+1} &= 7 \\ \underline{\cancel{-1}} &\quad \underline{-1} \end{aligned}$$

$$\frac{\cancel{2}n}{\cancel{2}} = \frac{6}{2}$$

$$n = 3$$

Checking:

$2(3) + 1 = 7$?
$6 + 1 = 7$?
$7 = 7$
✔ (this checks)

Example:

$$\begin{aligned} 8x \cancel{-2} &= 2 \\ \underline{\cancel{+2}} &= \underline{+2} \end{aligned}$$

$$\frac{8x}{8} = \frac{4}{8}$$

$$x = \frac{1}{2}$$

Participants will probably panic at the fractional answer, but they need to realize that not all answers are whole numbers or integers.

Checking:

$8(1/2) - 2 = 2$
$4 - 2 = 2$
$2 = 2$
✔ (this checks)

Participants should realize that algebra is very methodical and has several nonvarying rules. It is also important to note that memorization will not produce success. Problem solving involves knowing where to start and then making a plan.

The final activity practice will tie together all concepts from this module.

4.4 Activity: Tricky Translations

The paraeducator will use translation skills to create and solve basic equations.

Materials:
- Transparency/handout **Tricky Translations (T10/H10)**

4.4.1 Steps
- Remind participants that the word "is" signals the equals sign for the equation. Words to the left are on the left; words to the right are on the right.
- In pairs, have participants complete the handout **Tricky Translations (H10).**
- Go over answers as a group using the transparency **Tricky Translations (T10).**

1. A number multiplied by 3 is 21.

$n \bullet 3 = 21$

$$\frac{\cancel{3}n}{\cancel{3}} = \frac{21}{3}$$

$n = 7$

   - In algebra, the number (coefficient) is placed before the variable. While the problem solves the same, the format is different.

2. 69 plus a number is 152.

$69 + y = 152$

$$\begin{array}{rcl} 69 + y & = & 152 \\ \underline{-\cancel{69}} & & \underline{\cancel{-69}} \\ y & = & 83 \end{array}$$

   - Order does not matter in addition, so the problem may be written in any order ($y + 69 = 152$).

3. A number divided by 11 is -4.

$$\frac{z}{11} = -4$$

$$(\cancel{11})\frac{z}{\cancel{11}} = (-4)\ (11)$$

$$z = -44$$

- Order is important here. Watch the negative signs.

4. The difference between a number and 3 is 82.

$$p - 3 = 82$$
$$p \cancel{-3} = 82$$
$$\underline{\cancel{+3}} \quad \underline{+3}$$
$$p = 85$$

- When order for subtractions is not given, write in the order they occur.

5. 70 divided by a number is 10.

$$\frac{70}{n} = 10$$

$$(\cancel{n})\frac{70}{\cancel{n}} = (10)(n)$$

$$\frac{70}{10} = \frac{10n}{10}$$

$$7 = n$$

- This is a difficult problem because the variable is on the bottom.
- Encourage participants to solve this intuitively (answer for x is 7) first to see if they can see any patterns.
- Go back to problem #3 to note how to clear a denominator; apply the same principle.
- May also be set up and solved as a proportion if the group has the requisite prior knowledge to do so. This will be covered later in this Academy.

$$\frac{70}{n} = \frac{10}{1}$$

$$10 \cdot n = 70 \cdot 1$$

$$\frac{10n}{10} = \frac{70}{10}$$

$$n = 7$$

6. Challenge: The sum of 3 and 2 times a number is 11.

$$3 + 2c = 11$$

$$\cancel{3} + 2c = 11$$

$$\underline{\cancel{-3}} \qquad \underline{-3}$$

$$\frac{\cancel{2}c}{\cancel{2}} = \frac{8}{2}$$

$$c = 4$$

   - This is a multi-step equation. Remind participants to do the numbers furthest from the variable first.

- These examples do not cover all the problem types that are encountered in solving basic equations. This exercise is a start towards problem solving.

5.1 Assignment #2: Algebra Skills

Note to Instructor: Decide how long class members will have to complete their assignments so that you have time to grade, record grades and turn in materials from this Academy in a timely manner. If paraeducators are taking the Academy for credit, there will be a time limit based upon the grading period at the attending institution. You will also have to decide how you would like attendees to turn in their assignments to you. For example, they can be mailed or you can make whatever arrangements seem to work for you and your class. **You are strongly encouraged to be firm about the completion date and may need to make some effort to follow up on attendees and their progress**. **Refer to the Grading Rubric handout (GR) for details on grading.**

Distribute handout **Assignment #2: Algebra Skills (H11):** Read the instructions and answer questions regarding completion of the assignment. Provide the class with a date for completion and explain your process for handing in the assignment. **To assist in grading the assignment, answers to the questions are listed under each question. Please ensure that the answers are not released to the students before they complete the assignment.**

The following assignment is worth 125 points. The focus of this assignment is to practice all algebra skills learned in this module.

Translate the following into algebraic expressions. (25 points)

1. 48 divided by a number e **Answer:** $\frac{48}{e}$
2. 5 more than an unknown **Answer:** $5 + h$ or $h + 5$ (any variable)
3. the product of 7 and a number f **Answer:** $7f$

Evaluate the following expressions. (15 points)

4. If $n = 4$, then $6n = ?$

   Answer: $6(4) = 24$

5. If $q = -2$, then $2q + 4 = ?$

   Answer:
   $2q + 4 =$
   $2(-2) + 4$
   $-4 + 4 = 0$

Solve the following by inspection: (10 points)

6. $54 \div s = 9$ **Answer:** $s = 6$

7. $g - 4 = -3$ **Answer:** $g = 1$

Solve the following equations using algebra. Show all steps to receive full credit. Make sure to show how you checked your answers. (30 points)

8. $6 + y = 22$

   Answer:
   $\cancel{6} + y = 22$
   $\underline{\cancel{-6}} \quad \underline{-6}$
   $y = 16$

9. $x - 15 = -16$

   Answer:
   $x - \cancel{15} = -16$
   $\underline{+\cancel{15}} \quad \underline{+15}$
   $x = -1$

10. $-78 = -39p$

    Answer:
    $\frac{-78}{-39} = \frac{\cancel{-39}p}{\cancel{-39}}$
    $2 = p$

11. $\frac{m}{2} = 10$

    Answer:
    $\cancel{2} \bullet \frac{m}{\cancel{2}} = 10 \bullet 2$
    $m = 20$

12. Challenge: $2r + 2 = 6$

    Answer:
    $2r + 2 = 6$
    $\underline{-2} \quad \underline{-2}$
    $\frac{\cancel{2}r}{\cancel{2}} = \frac{4}{2}$
    $r = 2$

Translate and evaluate the following. Make sure to check your answers. (40 points)

13. A number divided by 4 is 3.

$$\frac{m}{4} = 3$$

Answer: $$\cancel{4} \cdot \frac{m}{\cancel{4}} = 3 \cdot 4$$

$$m = 12$$

14. The sum of a number and 4 is 15.

$$y + 4 = 15$$

Answer: $$\begin{aligned} y + \cancel{4} &= 15 \\ \underline{-\cancel{4}} \quad & \underline{-4} \\ y &= 11 \end{aligned}$$

15. Six times a number is 12.

$$6b = 12$$

Answer: $$\frac{\cancel{6}b}{\cancel{6}} = \frac{12}{6}$$

$$b = 2$$

Module C Handouts

What's Next?

Complete the missing terms in each pattern below. Make notes about the patterns that you see.

1. 5, 9, 13, 17, 21, ______, ______, ______, ______, …

   Pattern:

2. 13, 10, 7, 4, 1, ______, ______, ______, …

   Pattern:

3. 1, 3, 7, 13, 21, ______, ______, ______, ______, …

   Pattern:

4. 7, 14, 21, 28, ______, ______, ______, ______, …

   Pattern:

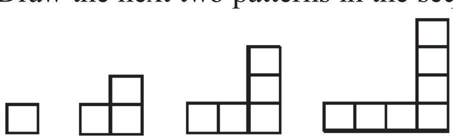
5. ______, 64, 32, 16, ______, ______, ______, ______,   , …

   Pattern:

6. ______, 6/5, 8/5, 2, 12/5, ______, ______, ______, 4, …

   Pattern:

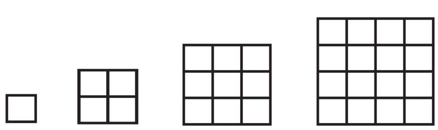
7. Draw the next two patterns in the sequence.



, ______________, ______________

   Pattern:

8. Draw the next two patterns in the sequence.



, ______________, ______________

   Pattern:

Types of Patterns

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Elements in the patterns repeat according to some rule. Repeating patterns are an easy place to begin helping students with prediction skills.

Example: circle, triangle, circle, triangle, _____

Growing patterns:

Elements in the patterns change (grow) according to some rule. These types of patterns are the introduction to algebraic concepts. The simplest growing pattern is counting.

Arithmetic sequence:

A sequence where each successive term is obtained from the previous term by the addition or subtraction of a fixed number, the **difference**.

Example: 1, 4, 7, 10, … The constant or **difference** is 3.

Geometric sequence:

A sequence where each successive term is obtained from its predecessor by multiplying or dividing by a fixed number, the **ratio**.

Example: 2, 4, 8, 16, 32, … The **ratio** is 2.

Keys to Algebra

Algebra:

Algebra is a method by which to describe and predict the behavior of a set of data. *Data* may be anything: blocks, toys, weather, numbers, etc.

A Variable:

- Symbol, object, or letter that can change value depending on the problem in which it appears
- Simply holds a place until is has a value, then it changes with the next problem
- Has a value that can vary, or is "variable"

Expression:

Rule or "phrase" that represents a pattern or generalization. Also called a function. Consists of variables, constants, numerals, and operation signs.

Addition (+)	**Subtraction** (-)	**Multiplication** (x)	**Division** (÷)
Add	Subtract	Multiply	Divide
Added to	Subtracted from	Multiplied by	Divided by
Sum	Difference	Product	Quotient
Total	Minus	Times	
Plus	Less than	Of	
More than	Decreased by		
Increased by	Take away		

Translating Algebraic Expressions:

- 4 <u>more than</u> p
- y <u>decreased by</u> 10
- 5 <u>less than</u> *m*
- the <u>quotient</u> of a number and 7
- the <u>product</u> of 6 and an unknown

Translating Expressions into Words:

- n + 4
- 6 – z
- $\frac{10}{y}$
- 8b
- 5x + 1

Evaluating Expressions:

Using the expression and substituting actual values for the variable

Substituting:

Replace the variable with a numerical value

d	*0.10d*	=
0	0.10 (0)	0
1	0.10 (1)	\$0.10
2	0.10 (2)	\$0.20
3	0.10 (3)	\$0.30
4	0.10 (4)	\$0.40
5	0.10 (5)	\$0.50
6	0.10 (6)	\$0.60

Example:

- If p = 0, then $2p + 3 =$
- If n = -2, then $-4n - 1 =$
- If n = 1, then $(16 \div 8) - n = ?$
- If a = 2, then $a^2 =$

Expressing Expressions

Write an algebraic expression to represent each scenario. Remember to define a variable and then decide what you would do in each situation to solve a real problem.

Example: Amount of money in dimes:

d = the number of dimes
5 dimes would be \$0.50 or 0.10 x 5
3 dimes would be \$0.30 or 0.10 x 3
7 dimes would be \$0.70 or 0.10 x 7
Expression: $0.10 \times d$ or $0.10d$

1. Amy's age in 5 years:

2. The number of hours remaining from a 12-hour driving trip:

3. The number of cookies for each person if 50 cookies were shared equally:

4. The number of game cards Matt and Rosie combined if Rosie has 20 cards:

5. The number of pictures in Juan's scrapbook if each page holds 10 pictures:

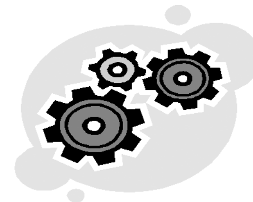
6. Let m equal the number of motorcycles in a parking lot and c equal the number of cars in a parking lot. Write an expression for each of the following:

    a. The total number of wheels on all the motorcycles in the parking lot

    b. The total number of wheels on all the cars in the parking lot

    c. The total number of wheels in the parking lot on all the cars and motorcycles

Guess My Rule



Complete the chart for each function machine. Pay close attention to the variable used for the input in order to create your rule. Remember to look carefully at what goes in and what comes out.

1. Rule: q – 10

Input (q)	26			66	73	74	
Output		29	45				83

2. Rule: ______________________________

Input (q)	9	17	25	33	41
Output	27	51	75	99	123

3. Rule: ______________________________

Input (r)	19	30		52	63
Output	30		52	63	69

4. Rule: ______________________________

Input (f)	15	20	25		35	40	45
Output	3		5		7	8	

Algebraic Patterns

nth term:
Asks for a general expression to predict any term of the sequence.

Sequence: 2, 4, 6, 8, …

Term number	1	2	3	4	5	6	7	n	100
Terms	2	4	6	8	10	12	14		

Sequence: 3, 5, 7, 9, …

Term Number	1	2	3	4	5	6	7	n	100
Terms	3	5	7	9	11	13	15		

Term Number	1	2	3	4	5	6	7
2*n*	2	4	6	8	10	12	14
Desired Terms	3	5	7	9	11	13	15

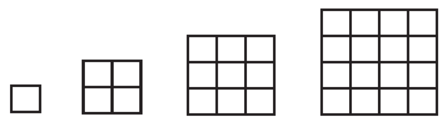
Guess My Rule II

Fill in the remaining numbers for the sequence. Find a rule for the *nth term.* Remember to look at the relationship between the term number and the sequence. Use the sequence as a hint to get you started. This time the term number is the input. Find the 100th term for each sequence.

1. *nth term:* ______________________________

Term Number (n)	1	2	3	4	5	6	7	...100
Sequence	7	14	21	28				

2. *nth term:* ______________________________

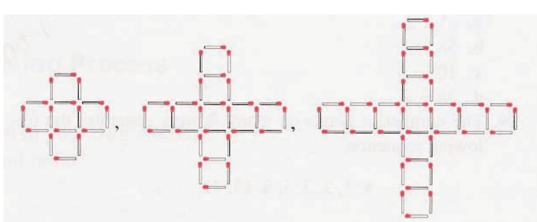


Term Number (n)	1	2	3	4	5	6	7	...100
Sequence	1	4	9	16				

3. *nth term:* ______________________________

Term Number (n)	1	2	3	4	5	6	7	...100
Sequence	5	9	13					

Matchsticks:



Solving Basics

Equation:

An expression (sentence) with an equals sign.

Solution:

A number that makes the equation true.

Equations differ from *expressions* in two major ways:

- Contain an equals sign
- The goal is to find all of the values for the variable that make the statement true

Goal of algebra:

Isolate the variable, which implies solving the equation.

Examples: Solve by inspection.

$x + 9 = 17$

$3y = 12$

$18 = m - 7$

$z \div 4 = 4$

Some general suggestions for solving:

- Show work on both sides
- Cross out concepts that cancel
- Keep equals signs aligned (in the vertical method)

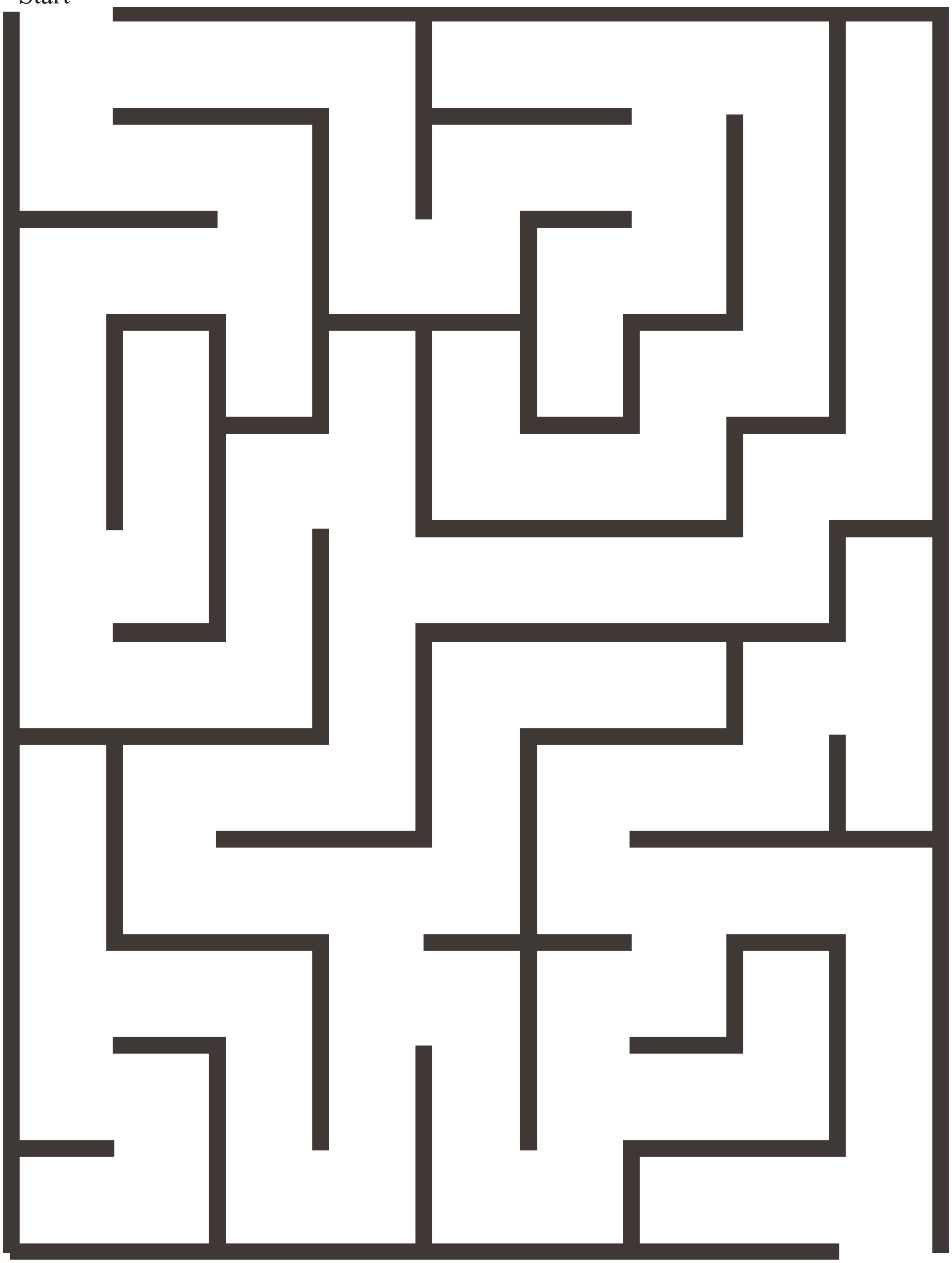
Multi-Step Examples:

$2n + 1 = 7$

$8x - 2 = 2$

Maze Mischief

Start



Finish

Tricky Translations

Translate each phrase into an algebraic equation. Solve each equation and make sure to check your answers.

1. A number multiplied by 3 is 21.

2. 69 plus a number is 152.

3. A number divided by 11 is - 4.

4. The difference between a number and 3 is 82.

5. 70 divided by a number is 10.

6. Challenge: The sum of 3 and 2 times a number is 11.

Assignment #2: Algebra Skills

The following assignment is worth 120 points.

The focus of this assignment is to practice all algebra skills learned in this module.

Translate the following into algebraic expressions. (25 points)

1. 48 divided by a number e

2. 5 more than an unknown

3. the product of 7 and a number f

Evaluate the following expressions. (15 points)

4. If $n = 4$, then $6n = ?$

5. If $q = -2$, then $2q + 4 = ?$

Solve the following by inspection: (10 points)

6. $54 \div S = 9$

7. $g - 4 = -3$

Solve the following equations using algebra. Show all steps to receive full credit. Make sure to show how you checked your answers. (30 points)

8. $6 + y = 22$

9. $x - 15 = -16$

10. $-78 = -39p$

11. $\frac{m}{2} = 10$

12. Challenge: $2r + 2 = 6$

Translate and evaluate the following. Make sure to check your answers. (40 points)

13. A number divided by 4 is 3.

14. The sum of a number and 4 is 15.

15. Six times a number is 12.

Module C
Transparencies

Module C: Algebraic Fundamentals

The paraeducator will:

- **Use patterns and sequences to predict and generalize outcomes**
- **Describe patterns and other relationships using words and expressions**
- **Relate basic patterns to algebraic concept development**
- **Develop a plan for solving basic algebraic equations**

Types of Patterns

Repeating patterns:

Elements in the patterns repeat according to some rule. Repeating patterns are an easy place to begin helping students with prediction skills.

Examples: circle, triangle, circle, triangle

Growing patterns:

Elements in the patterns change (grow) according to some rule. These types of patterns are the introduction to algebraic concepts. The simplest growing pattern is counting.

Types of Patterns

Arithmetic sequence:

A sequence where each successive term is obtained from the previous term by the addition or subtraction of a fixed number, the **difference**.

Example: 1, 4, 7, 10, …

The constant or **difference** is 3.

Geometric sequence:

A sequence where each successive term is obtained from its predecessor by multiplying or dividing by a fixed number, the **ratio**.

Example: 2, 4, 8, 16, 32, …

The **ratio** is 2.

Math Words

-
-
-
-
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Keys to Algebra

Algebra:

Algebra is a method by which to describe and predict the behavior of a set of data. *Data* may be anything: blocks, toys, weather, numbers, etc.

A Variable:

- Symbol, object, or letter that can change value depending on the problem in which it appears
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- Has a value that can vary, or is “variable”

Expression:

Rule or “phrase” that represents a pattern or generalization. Also called a ***function.*** Consists of variables, constants, numerals, and operation signs.

Keys to Algebra

Addition (+)	Subtraction (-)	Multiplication (x)	Division (÷)
Add	Subtract	Multiply	Divide
Added to	Subtracted from	Multiplied by	Divided by
Sum	Difference	Product	Quotient
Total	Minus	Times	
Plus	Less than	Of	
More than	Decreased by		
Increased by	Take away		

Translating Algebraic Expressions:

- 4 more than *p*
- y decreased by 10
- 5 less than *m*
- the quotient of a number and 7
- the product of 6 and an unknown

Keys to Algebra

Translating Expressions into Words:

- n + 4
- 6 – z
- $\frac{10}{y}$
- $8b$
- $5x + 1$

Keys to Algebra

Evaluating Expressions:

Using the expression and substituting actual values for the variable

Substituting:

Replace the variable with a numerical value

d	$0.10d$	=
0	0.10 (0)	0
1	0.10 (1)	$0.10
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3	0.10 (3)	$0.30
4	0.10 (4)	$0.40
5	0.10 (5)	$0.50
6	0.10 (6)	$0.60

Keys to Algebra

Example:

If $p = 0$, then $2p + 3 = ?$

If $n = -2$, then $-4n - 1 = ?$

If $n = 1$, then $(16 \div 8) - n = ?$

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Expressing Expressions

Write an algebraic expression to represent each scenario. Remember to define a variable and then decide what you would do in each situation to solve a real problem.

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7 dimes would be \$0.70 or 0.10 x 7

Expression: $0.10 \times d$ or $0.10d$

1. Amy's age in 5 years:

   A = Amy's age now

   A + 5 Amy's age in 5 years

2. The number of hours remaining from a 12-hour driving trip:

   h = # of hours driven

   12 – h Numbers of hours remaining

3. The number of cookies for each person if 50 cookies were shared equally:

   C = # of people at the party

   $50/c$ Number of cookies for each person

Expressing Expressions

4. The number of game cards Matt and Rosie combined if Rosie has 20 cards:

   m = # of cards Matt has

   $20 + m$ Number of cards they have total

5. The number of pictures in Juan's scrapbook if each page holds 10 pictures:

   p = # of pages in the scrapbook

   $10p$ Number of pictures in the book

6. Let m equal the number of motorcycles in a parking lot and c equal the number of cars in a parking lot. Write an expression for each of the following:

   a. The total number of wheels on all the motorcycles in the parking lot

      $2m$ Each motorcycle has 2 wheels

   b. The total number of wheels on all the cars in the parking lot

      $4c$ Each car has 4 wheels

   c. The total number of wheels in the parking lot on all the cars and motorcycles

      $2m + 4c$ Total for all wheels

Guess My Rule

Complete the chart for each function machine. Pay close attention to the variable used for the input in order to create your rule. Remember to look carefully at what goes in and what comes out.

1. Rule: q – 10

Input (q)	26			66	73	74	
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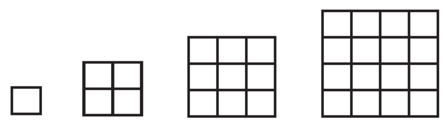

Term Number (n)	1	2	3	4	5	6	7	...100
Sequence	1	4	9	16				

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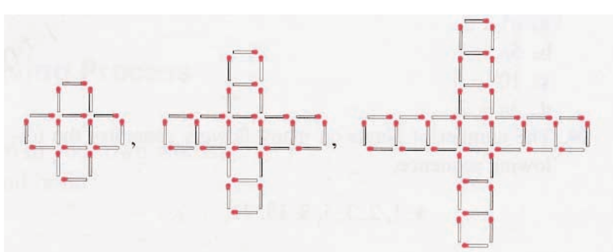
Term Number (n)	1	2	3	4	5	6	7	...100
Sequence	5	9	13					

Matchsticks:



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$z \div 4 = 4$

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General suggestions for solving:

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Multi-Step Examples:

$2n + 1 = 7$

$8x - 2 = 2$

Tricky Translations

Translate each phrase into an algebraic equation. Solve each equation and make sure to check your answers.

1. A number multiplied by 3 is 21.

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Tricky Translations

4. The difference between a number and 3 is 82.

5. 70 divided by a number is 10.

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Module D
Instructor's Guide

Module D: Graphic Representations

A. Module Goals
Using the transparency **Module Goals (T1),** review the goals for Module D.

Module D: Graphic Representations
The paraeducator will:
1. Explore linear and nonlinear functions as they represent data patterns
2. Interpret linear graphs as rates of change (i.e., slope)
3. Sketch graphs that represent real-life situations

Graphs are a common way to visually represent data. Graphs are present in everyday life when we watch the news or read a newspaper. They are helpful in identifying patterns in data that may not be visible by looking at raw number data.

A common error with algebra and graphing is to jump into the mathematical formulas too early before students learn to interpret the data from graphs. Many students do not make the link between data and graphs until physics or calculus because the emphasis was put on computing formulas and generating graphs. Reversing that logic makes students more astute to looking for patterns as they work with data and more likely to question odd results.

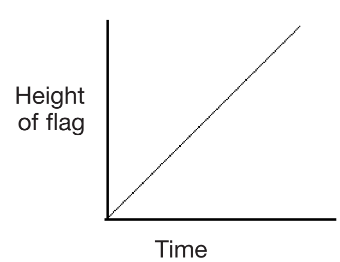
Goal 1: Explore linear and nonlinear functions as they represent data pattern.

1.1 Discussion: Interpreting Graphs

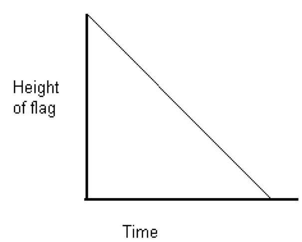
The paraeducator will use graphs to interpret data.

Materials:
- Transparency/handout **Seeing Graphs (T2/H1)**

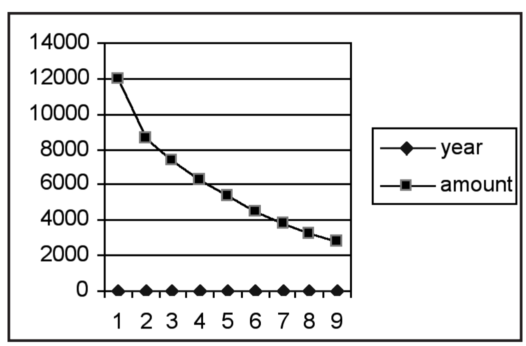
1.1.1 Steps
- Graphs come in all shapes; many graph shapes can be linked to specific kinds of patterns.
- The first graph shown represents hoisting a flag. Ask the following questions (see the transparency/handout **Seeing Graphs [T2/H1]**):



▲ How would you describe the x axis? (represents time as it increases from left to right)
▲ How would you describe the y axis? (represents the height of a flag as it increases from 0 going up)
▲ How would you give a scenario for this graph? (the flag starts at the ground and increases in height as the time increases)
▲ What can be said about the rate of height increase to time increase? (the height increases at a constant rate over time because the line is straight)
▲ What would the graph look like if the flag were being lowered? Ask participants to sketch their guess making sure to label the axes. (the line would be reversed. It would start at the maximum height and decrease as time increased.)
  o Draw the graph as participants describe it. Be very specific about where the graph starts and ends.
  o Stop drawing the graph before getting to zero. Ask what this represents. (the flag is still on the flagpole)



• The second graph represents the value of a used car. Ask the following questions:



▲ Does this graph represent appreciation or depreciation? (depreciation)
▲ How do we know? (assume the y axis is a dollar amount and the x axis is years owned. Price decreases over time)
▲ What would make this easier to understand? (labeling the axes)
  o Label x axis: years owned
  o Label y axis: car value in dollars
▲ Do we know what the original price of the car actually was? (no, not from this graph as it does not start at zero time)
▲ Where is the largest drop in value on this graph? (year 1 to year 2)

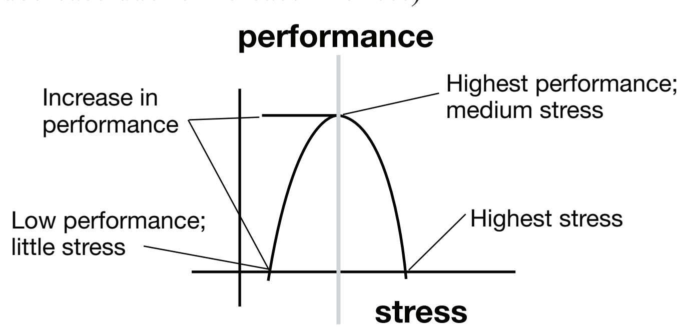
Note to Instructor: The largest drop occurs when the car is new the first year. This graph assumes that the used car is one year old.

- ▲ What happens after the initial large drop according to the graph? (the depreciation slows down)
- ▲ How would you explain the trend? (the car is worth less each year, so less money is lost each year)

- The third graph shows stress and performance. Allow participants to answer these questions individually; then discuss as a group (see the transparency **Seeing Graphs [T2]** for the complete scenario).

Note to Instructor: Performance is the label for the y axis, not the title of the graph. This label would typically be placed on the left side of the axis rather than on top as the original author chose.

1. Write a statement that describes performance as stress increases. (at the start, performance increases as stress increases. There is a maximum value at which performance starts to decrease as stress continues to increase)
2. Which part of the graph illustrates where *stress* is highest? (stress is the highest where the right leg of the graph touches the axis)
   - o A common error is to say the top of the graph is maximum stress. Remind participants to look at the axis labels
3. What is *performance* at that point? (performance is at or near zero)
4. Which part of the graph illustrates where *performance* is highest? (at the top of the arc)
5. Which part of the graph illustrates where *performance* is increasing? (The initial rise of the graph to the top of the arc)
6. Which part of the graph illustrates where *performance* is decreasing? (the fall after the arc on the right)
7. Notice that the graph is symmetric about a vertical line. What would you say this indicates about performance? (there is a definite point half way through where performance begins to decrease due to increase in stress)



1.2 Lecture: Types of Graphs

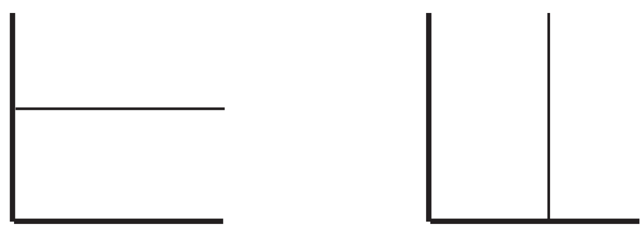
***Note to Instructor**:* Remind participants to take notes during lecture periods.

The examples on the handout **Seeing Graphs (H1)** introduced several graphing concepts. An important place to start is to classify graph types into two simple categories: *linear* and *nonlinear*.

Looking at the term *linear*, the base word is *line*. That is, a *linear graph* is a line. This matches the graph example for Hoisting a Flag. By definition, data are *linearly related* if the graph increases or decreases at a *constant rate*. From an equation, we can tell if it is linear if the exponent on the variable is 1 (see the transparency/handout **Graphing Basics (T3/H2).**

$y = 4x$
$y = -2x + 1$

Have participants sketch examples of the linear graphs including Hoisting the Flag and the following sketches on the transparency/handout **Graphing Basics (T3/H2)**. Note that the line may have any orientation.

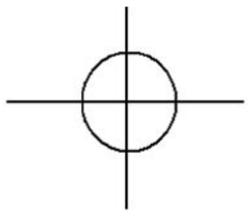


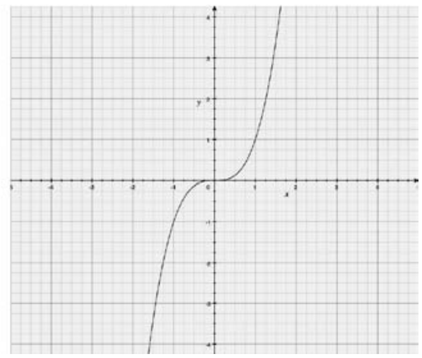
Nonlinear graphs do not make a line. Car Values and Stress and Performance are examples of *nonlinear* graphs. This implies that the data do not increase or decrease at a constant rate. There are many subcategories of *nonlinear* graphs (these are covered in the upper grades). Mathematically, nonlinear graphs may be determined from equations by looking at the highest exponent. Nonlinear equations have exponents that are second degree (2) or higher.

$y = x^2 + 1$
$y = x^2 + 4x + 4$
$y = x^3 – 6$

Have participants sketch examples of nonlinear graphs including Car Values, Stress and Performance, and the following sketches.

Note to Instructor: Car Values is an *exponential*. It will continue to decrease gradually in a smooth decreasing arc as it approaches zero. Be careful not to sketch it as a straight line.





Goal 2: Interpret linear graphs as rates of change (i.e., slope).

2.1 Activity: Slippery Slopes

The paraeducator will define slope from graphing examples.

Materials:
- Transparency/handout **Slippery Slopes (T4/H3)**
- Transparency **Slippery Slopes Answers (T5)**
- Ruler or straightedge

2.1.1 Steps
- As graphs are drawn on coordinate axes, ordered pairs that come from data are necessary to create the graphs.
- The ordered pairs come from the input and output data of patterns and sequences. The x is still the input value and y is the output value, also called the *function value*.
- Complete the first table/line on the transparency/handout **Slippery Slopes (T4/H3)** as a group to follow the process for drawing a linear graph listed on the transparency/handout **Graphing Basics (T3/H2)**.
  - ▲ Complete the input/output table by substituting x values into the equation (function) and evaluating
  - ▲ Plot the points on the graph
  - ▲ Connect the points. Make sure to trace the line completely through and place arrowheads on the line to show a continuous line
  - ▲ Label the line with its equation
- In pairs, have participants complete Part I of the handout **Slippery Slopes (H3)**.
- Check participants' tables while walking around the room. Draw the graphs on the transparency **Slippery Slopes (T4)** to check the placement of the lines (the answers are listed on the transparency **Slippery Slopes Answers [T5]**). Do not use triangles for this exercise.

Note to Instructor: There are many formulas and shortcuts for drawing lines. These concepts will be reserved for higher grades. Basic graphing ideas are the emphasis in this goal.

- Before moving on to the lecture, briefly discuss the results:
  - ▲ If you walked from left to right on each graph, which would be the steepest walk? ($y = 3x$)
  - ▲ Explain your decision. (line increases the fastest)
  - ▲ Which would be the easiest walk? ($y = 1/2x$)
  - ▲ Explain your decision. (line is gradual)

2.2 Lecture: Defining Slope

Note to Instructor: Remind participants to take notes during lecture periods.

In Colorado and other mountainous areas, you will encounter signs that read “Caution: Steep Grade 8%.” Truck drivers know that 8% means slow going up and potentially fast, or out-of-control, coming down.

Returning to the definition of percent, 8% means 8/100. In context of roads, it means that for every 100 ft a vehicle travels forward, the road increases in height by 8 feet. This relationship is a *ratio. Ratios* are used to compare data. We are familiar with *ratios* from concepts such as miles per hour when driving or dollars per pound when buying fruits or vegetables. Another name for ratio is *rate*. *Rate* measures how fast a value is changing (see the transparency/handout **Graphing Basics [T3/H2]**).

When discussing linear graphs and looking at whether the line is steep or gradual, the related mathematical concept is *slope*. The concept of *slope* consists of looking at the amount of “rise” over the amount of “run.” In mathematical terms, *slope* is concerned with the change in distance in the y direction to the change in distance in the x direction; hence, “rise” over “run” is written:

$$\frac{\Delta y}{\Delta x} = \frac{rise}{run}$$

The value of the *slope* may be positive (rising), negative (falling), zero (horizontal), or undefined (vertical).

***Note to Instructor*:** The formula above is commonly written in middle grades as $m = \Delta y/\Delta x = rise/run$, where m is the symbol that represents slope. The m is present in several general formulas for linear graphs. It is commonly associated with the slope-intercept formula of $y = mx + b$, which is used as a graphing shortcut. The information here is given to teach the concept of slope, and the formulas will not be included in this Academy.

2.3 Activity: Slippery Slopes (cont.)

The paraeducator will define slope from graphing examples.

Materials:
- Transparency/handout **Slippery Slopes (T4/H3)**
- Transparency **Slippery Slopes Answers (T5)**
- Ruler or straightedge

2.3.1 Steps
- Use the transparency **Slippery Slopes Answers (T5)** as a demonstration of using triangles to determine slope for $y = 3x$.
  - ▲ Find two points that occur exactly on an intersection on the grid: Use (2, 6) as point A and (4,12) as point B
  - ▲ From A to B, determine the change in "rise" to "run"
  - ▲ Draw a horizontal line from point A and a vertical line from point B until they intersect
  - ▲ Label the length of each side
  - ▲ Create the ratio of $\frac{rise}{run} = \frac{6}{2} = 3$
  - ▲ Choose another set of points and follow the same process
    - o (1,3) and (2,6)
    - o $\frac{rise}{run} = \frac{3}{1} = 3$
- Point out that any two points on this line will produce the same result. The *slope* of this line is 3. This is the constant increase that defines the line.
- Ask participants to find the slope of the other two lines to complete Part II of the handout **Slippery Slopes (H3).**
  - ▲ The slope of $y = \boldsymbol{x}$ is 1. (fractions will need to be reduced)
    - o Suggested points: (6,6) and (4,4), (2,2) and (1,1)
    - o Any two points any distance apart will provide a slope of 1
  - ▲ The slope of $y = 1/2\boldsymbol{x}$ is 1/2 (fraction will need to be reduced)
    - o Suggested points: (4,2) and (6,3), (1,1) and (4,2)
- Ask participants to complete Part III of the handout **Slippery Slopes (H3)** (see the transparency **Slippery Slope Answers [T5]** for correct tables and graphs).

**Note to Instructor:* The triangles may be drawn in any direction (above or below the line). The points should still be compared from left to right. Pay attention to the direction moved to get from one point to the other using the triangle. Left or down are negative. Right or up are positive.
  - ▲ The slope of $y = -2x$ is -2
    - o Suggested points: (-4, 8) and (-3, 6), (-3, 6) and (-1, 2)
  - ▲ The slope of $y = -x$ is -1
    - o Suggested points: (-2,2) and (-1,1), (-5,5) and (-2,2)
    - o Any combination of points works here. Fractions need to be reduced

2.4 Discussion: Slope Sense

The paraeducator will interpret slope data to identify key graphing information.

Materials:
- Handout **Slippery Slopes (H3)**

2.4.1 Steps:
- As mentioned throughout these modules, the ability to generalize patterns is the key to mathematical success.
- From the handout **Slippery Slopes**, ask the following questions:
  - ▲ For any two points on the same line, what was the resulting slope? (the resulting slope was always the same no matter what two points were chosen)
  - ▲ How do the slopes relate to the original functions (equations)? (the slopes are always the number next to the variable)
  - ▲ How does steepness relate to the value of Part II? (the line with the largest slope was the steepest line)
  - ▲ How does steepness relate to the value of Part III? (the negative just tells that the line is falling. Looking at the number tells which line is steeper. The line $y = -2x$ is steeper than $y = -x$.
    - o This can be confusing, as -1 is technically larger than -2. It is important to note that the negative just tells direction.
  - ▲ Is there a way to tell which way the line will go? (yes, if the slope is positive, the line increases from left to right. If the slope is negative, the line will decrease from left to right)
- Explain that the slope can always be found next to the variable. This number is called the *coefficient* in an algebraic equation.
- Point out that by looking at the value of the *slope*, we know whether the graph will *rise* (increase from left to right) or *fall* (decrease from left to right).
- Ask the following (you may want to return to the original sketches for linear graph and include slope data):
  - ▲ What is the slope of a *rising* line? (positive)
  - ▲ What is the slope of a *falling* line? (negative)
  - ▲ What would a *zero* slope imply? (this means that the line would not increase or decrease)
  - ▲ What would a line look like that had a *zero* slope? (it would be a horizontal line)
  - ▲ What would be the slope of a vertical line? (this slope has only vertical change. As a ratio, it would have a zero in the denominator which would mean that the fraction was undefined. The slope of a vertical line is *undefined*)

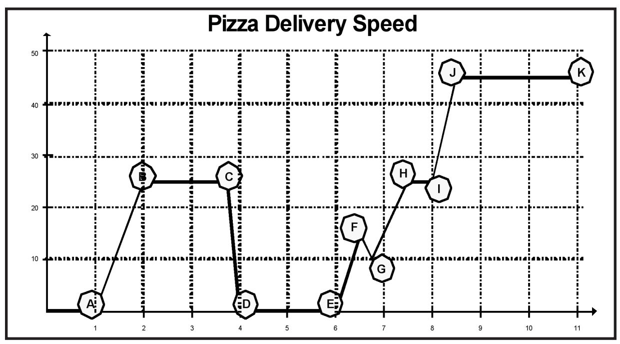
Goal 3: Sketch and interpret graphs that represent real-life situations.

3.1 Activity: Read All About It

The paraeducator will use slope to sketch and interpret graphs that represent real-life situations.

Materials:
- Transparency/handout **Read All About It (T6/H4)**
- Transparency/handout **Centimeter Grid Paper (T7/H5)**
- Blank transparencies (optional)

3.1.1 Steps
- Work in pairs or small groups to complete the handout **Read All About It (H4)**.
- Answers (use the transparency **Read All About It [T6]** as necessary):

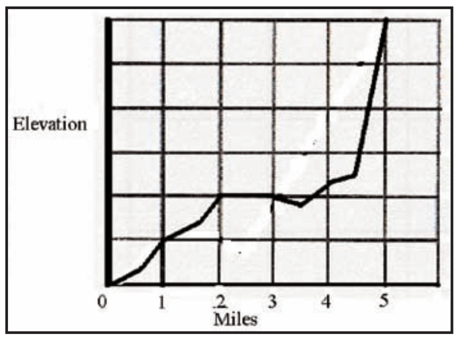


Part One
1. For what values of time (t) does the delivery car increase its speed? (AB, EF, GH, and IJ)
   - o Participants should be looking for increasing lines.
2. For what values of t did the pizza delivery car stop to deliver a pizza? (DE)
   - o Some participants will make the mistake of saying BC or JK since the line is flat. A zero speed (stop) would be at the bottom of the graph.
3. At what time does the delivery car achieve its highest speed? (JK)
4. For what values of t did the driver slow down to make a turn onto a street? (FG)
   - o Participants should be looking for a decreasing line but one that is fairly short to show a slight decrease in speed.
5. For what values of t does the car drive at a constant rate? (BC, HI, JK)
   - o Participants should be looking for flat lines that show constant speed over time.
6. At what values of t might the driver have come to a quick stop as he almost passed the house he was looking for? (CD)
   - o The key part is coming to a stop or speed of zero. CD shows a fast stop.

Part Two

The trailhead is considered the start of the 5-mile hike. The trail went up gradually for 2 miles. Between miles 2 and 3, the trail flattened out. For the next half mile, the trail went slightly downhill. For the next mile, it increased gradually, becoming very steep for the last half mile to the top of the mountain.

- Participants' graphs will vary for this problem, but they should have the same basic shape.
- Participants should agree that the graph increases from left to right.
- The first gradual incline between miles 0 and 2 should not be a single straight line as it is not a gradual slope.
- The sharpest incline (or largest slope) should be from 4.5 – 5 miles.



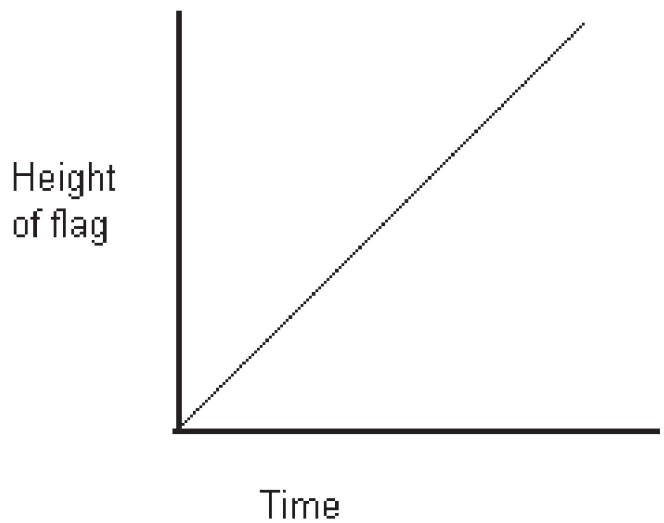
Part Three

This question is rather open-ended. Have participants check each other's work and clarify any questions. Share as many examples as time allows.

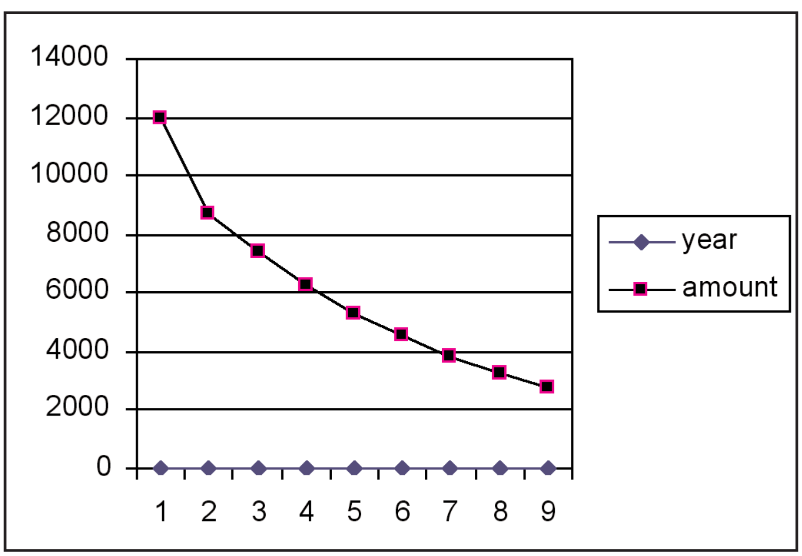
Module D
Handouts

Seeing Graphs

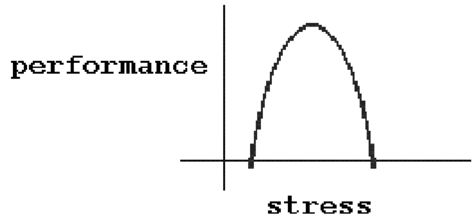
Hoisting a Flag



Value of Used Car



Stress and Performance



You have been training with a coach to run your first marathon. You are tense and not sure if you will be able to finish.

Your coach says, "Now, I know you are nervous. That's okay. It's good for you. You're going to perform better during the race if you're a little nervous." Does the graph above confirm what your coach told you?

Here are some questions based upon the graph. Write your answers or mark the graph above.

1. Write a statement that describes performance as *stress* increases.

2. Which part of the graph illustrates where *stress* is highest?

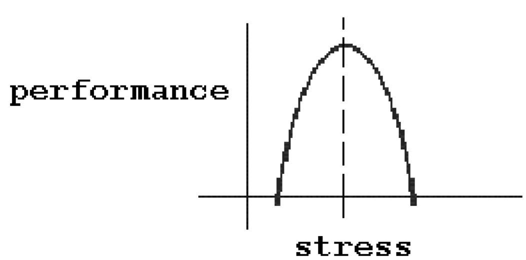
3. What is *performance* at that point?

4. Which part of the graph illustrates where *performance* is highest?

5. Which part of the graph illustrates where *performance* is increasing?

6. Which part of the graph illustrates where *performance* is decreasing?

7. Notice that the graph is symmetric about a vertical line. What does this indicate about performance?



Graphing Basics

Linear:

- Graph is a line
- Data increase or decrease at a constant rate
- Highest exponent on a variable is 1

Nonlinear:

- Graph does not make a line
- Data do not increase or decrease at a constant rate
- Graph may take many shapes
- Highest exponent on a variable is greater than 1

***Drawing a linear graph*:**

- Complete the input/output table by substituting x values into the equation (function) and evaluating
- Plot the points on the graph
- Connect the points. Make sure to trace the line completely through and place arrowheads on the line to show a continuous line
- Label the line with its equation

Ratio:

Comparison of two values

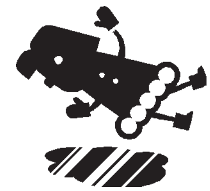
Example: miles per hour
dollars per pounds

Rate:

Same as a ratio. Shows how fast a value is changing.

Slope:

- Ratio or rate
- Concerned with the steepness of a line
- Measures the *change in y* to the *change in x* $\frac{\Delta y}{\Delta x} = \frac{rise}{run}$
- Can be positive (rising), negative (falling), zero (horizontal), or undefined (vertical)
- Found as the number next to the variable (*coefficient)* in the original equation



Slippery Slopes

Part I: Complete each table and plot the points for each linear equation. Connect the points to make a line for each table.

1.

y = *x*	
x	**y**
-1	
0	
1	
2	
4	
6	
10	

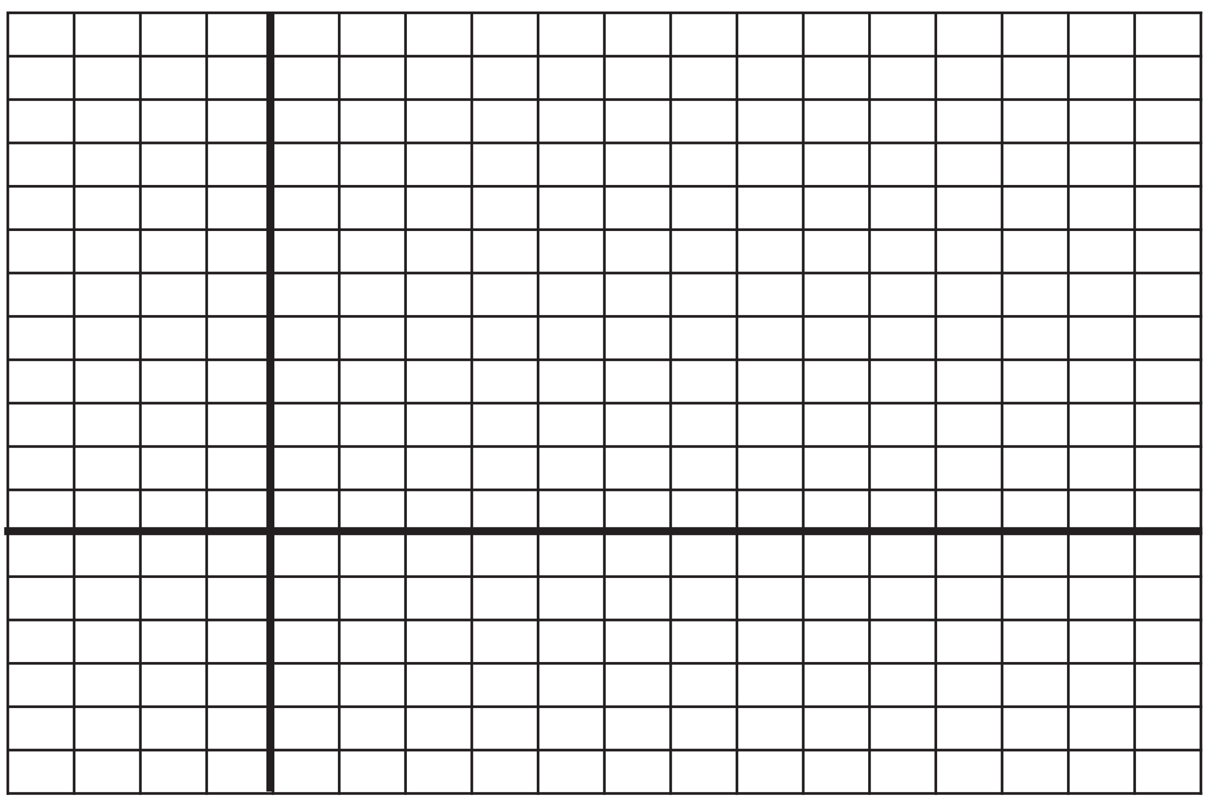
2.

y = 3*x*	
x	**y**
-1	
0	
1	
2	
3	
4	

3.

y = 1/2*x*	
x	**y**
-1	
0	
2	
4	
6	
8	
10	

Part II: Determine the slope of each line.



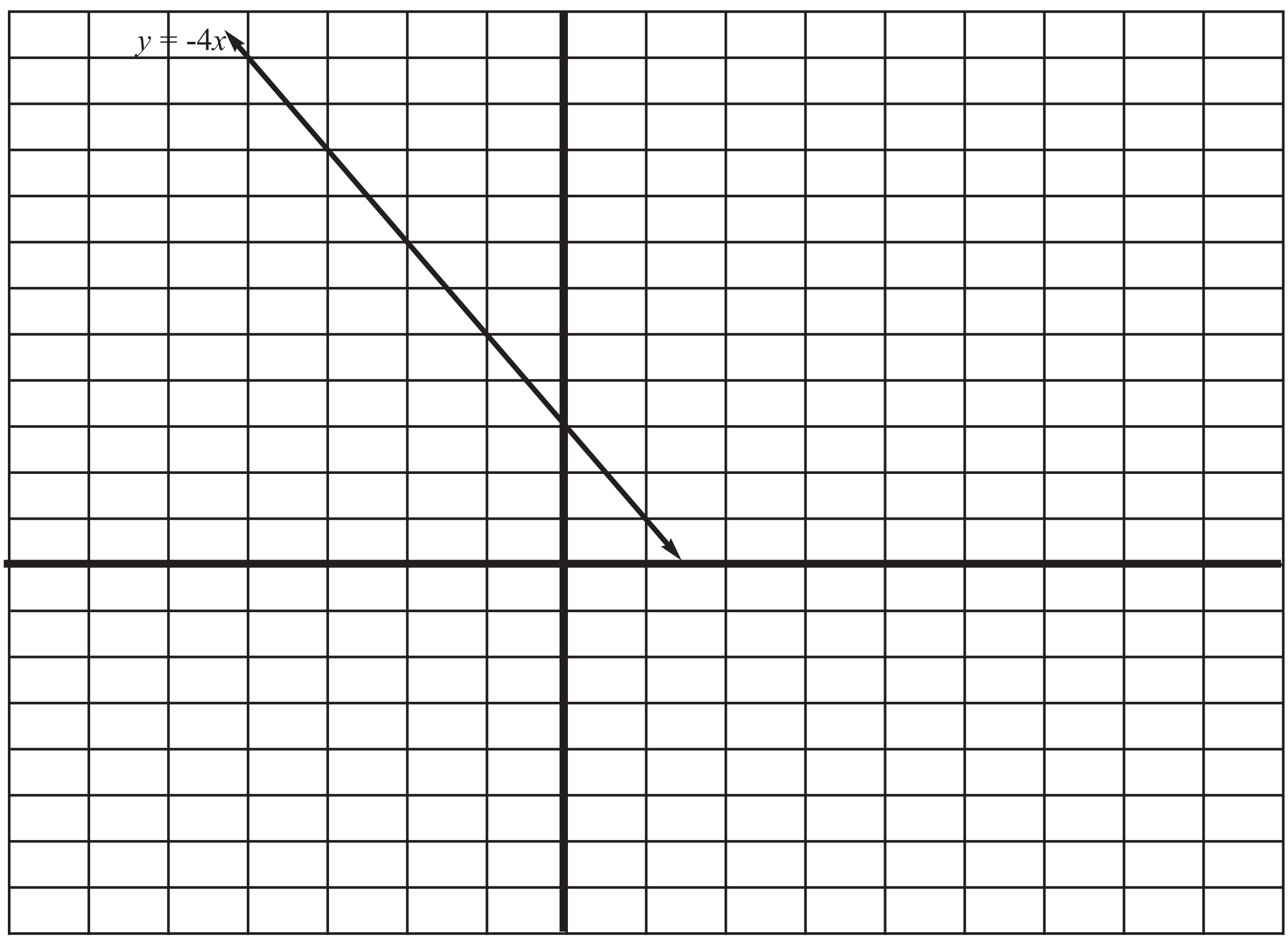
Part III:
One table and line have been completed. Complete the second table and line. Find the slopes of each line.

4.

$y = -2x$	
x	**y**
-4	8
-3	6
-2	4
-1	2
0	0
1	-2

5.

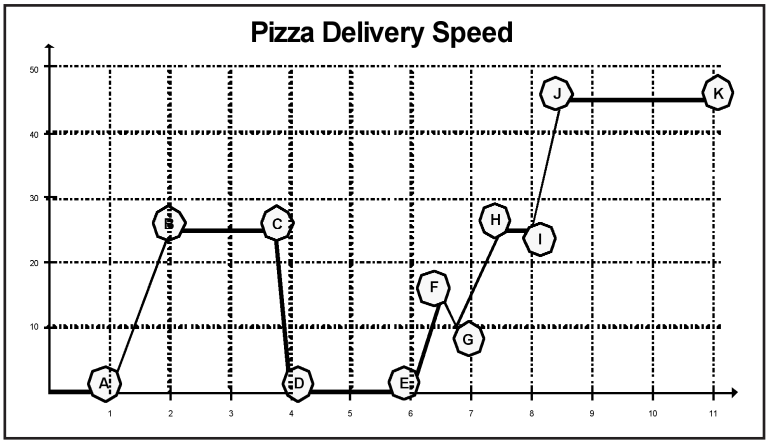
$y = -x$	
x	**y**
-4	
-3	
-2	
-1	
0	
1	



Read All About It

Use your knowledge of slope to complete and translate these stories.

1.



Part I

Study the graph above showing the activity of a pizza delivery car. Answer the following questions with values of time (*t*). (For example, AB includes all t between A and B).

1. For what values of time (*t*) does the delivery car increase its speed?

2. For what values of *t* did the pizza delivery car stop to deliver a pizza?

3. At what time does the delivery car achieve its highest speed?

4. For what values of *t* did the driver slow down to make a turn onto a street?

5. For what values of *t* does the car drive at a constant rate?

6. At what values of *t* might the driver have come to a quick stop as he almost passed the house for which he was searching?

Part II
Use centimeter graph paper to sketch a graph that represents the following description for a hike going up a mountain for 5 miles. Make sure to label your axes.

The trailhead will be considered the start of the 5-mile hike. The trail went up gradually for two miles. Between miles 2 and 3, the trail flattened out. For the next half mile, the trail went slightly downhill. For the next mile it increases gradually becoming very steep for the last half mile to the top of the mountain.

Part III
Write a scenario and draw its graph. Share your graph with a partner or group to check for its accuracy.

Centimeter Grid Paper

Module D
Transparencies

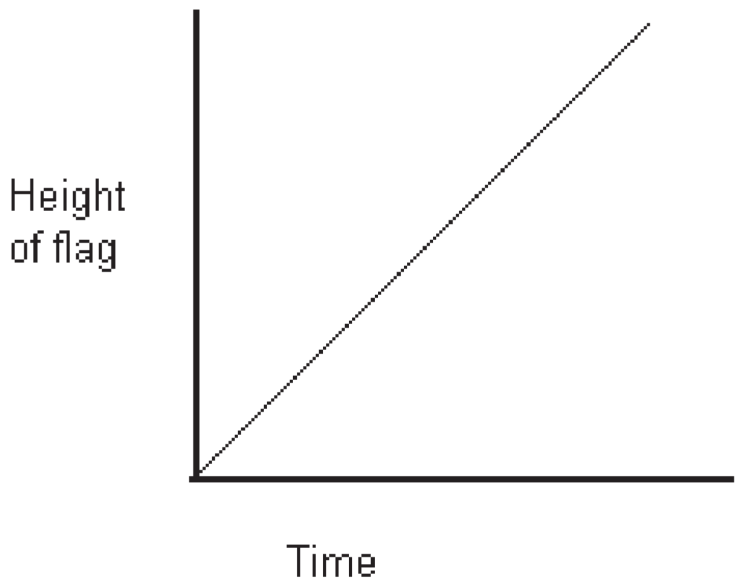
Module D: Graphic Representations

The paraeducator will:

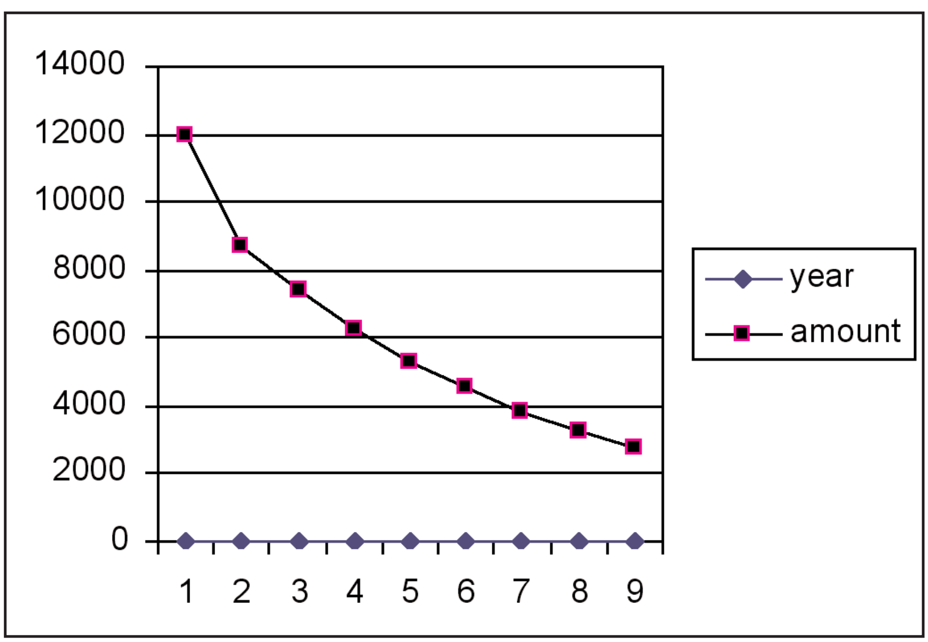
- Explore linear and nonlinear functions as they represent data patterns
- Interpret linear graphs as rates of change (i.e., slope)
- Sketch and interpret graphs that represent real-life situations

Seeing Graphs

Hoisting a Flag

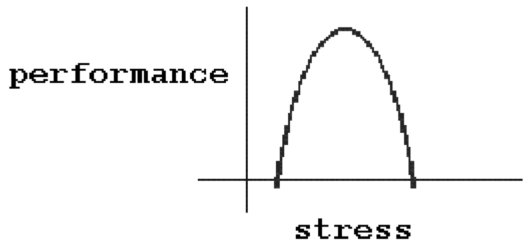


Used Car Value



Seeing Graphs

Stress and Performance



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Your coach says, "Now, I know you are nervous. That's okay. It's good for you. You're going to perform better during the race if you're a little nervous." Does the graph above confirm what your coach told you?

Here are some questions for you based upon the graph. Write your answers or mark the graph above.

1. Write a statement that describes performance as *stress* increases.

Seeing Graphs

2. Which part of the graph illustrates where *stress* is highest?

3. What is *performance* at that point?

4. Which part of the graph illustrates where *performance* is highest?

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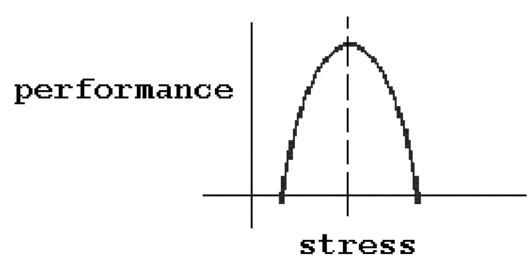
7. Notice that the graph is symmetric about a vertical line. What would you say this indicates about performance?



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- Highest exponent on a variable is greater than 1

Graphing Basics

***Drawing a linear graph*:**

- Complete the input/output table by substituting x values into the equation (function) and evaluating
- Plot the points on the graph
- Connect the points. Make sure to trace the line completely through and place arrowheads on the line to show a continuous line
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***Ratio*:**

Comparison of two values.

Example: miles per hour
dollars per pounds

Rate:

Same as a ratio. Shows how fast a value is changing.

Graphing Basics

Slope:

- Ratio or rate
- Concerned with the steepness of a line
- Measures the *change in y* to the *change in x*

$$\frac{\Delta y}{\Delta x} = \frac{rise}{run}$$

- May be positive (rising), negative (falling), zero (horizontal), or undefined (vertical)
- Found as the number next to the variable (*coefficient)* in the original equation

Slippery Slopes

Part I: Complete each table and plot the points for each linear equation. Connect the points to make a line for each table.

1.

$y = x$	
x	**y**
-1	
0	
1	
2	
4	
6	
10	

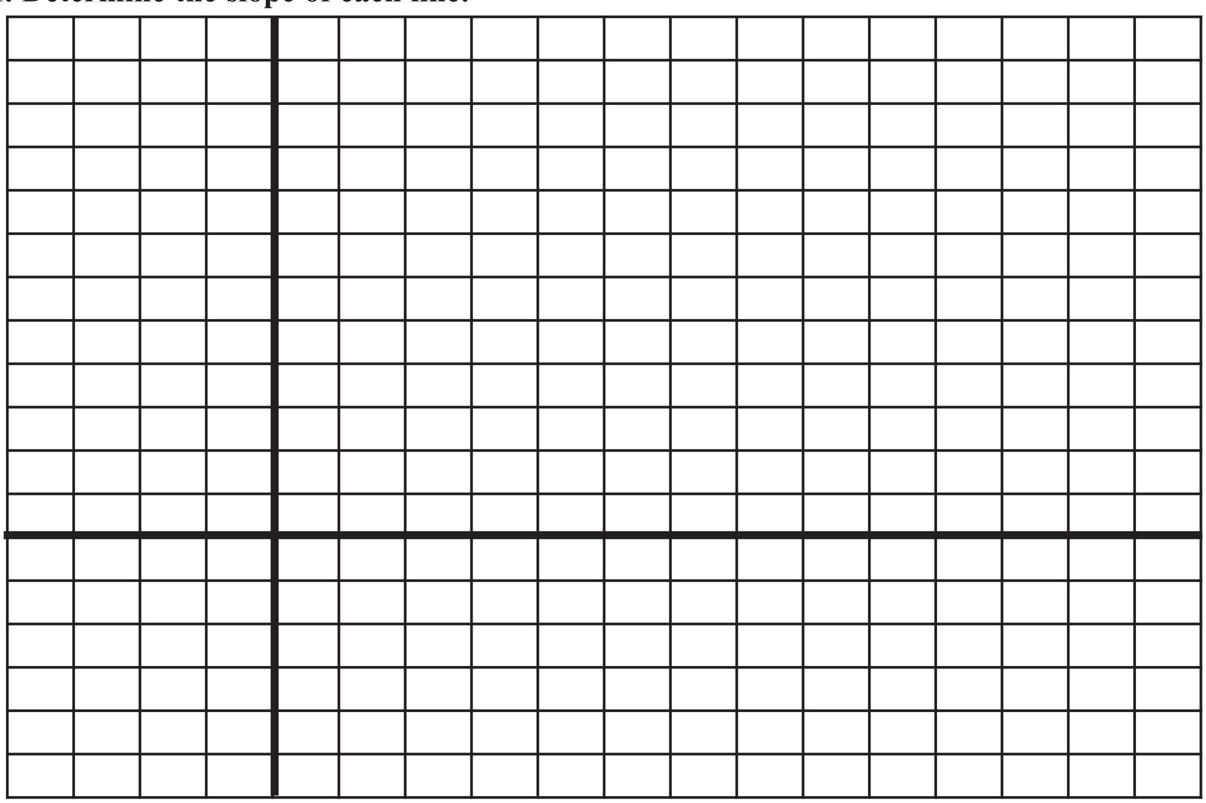
2.

$y = 3x$	
x	**y**
-1	
0	
1	
2	
3	
4	

3.

$y = 1/2x$	
x	**y**
-1	
0	
2	
4	
6	
8	
10	

Part II: Determine the slope of each line.



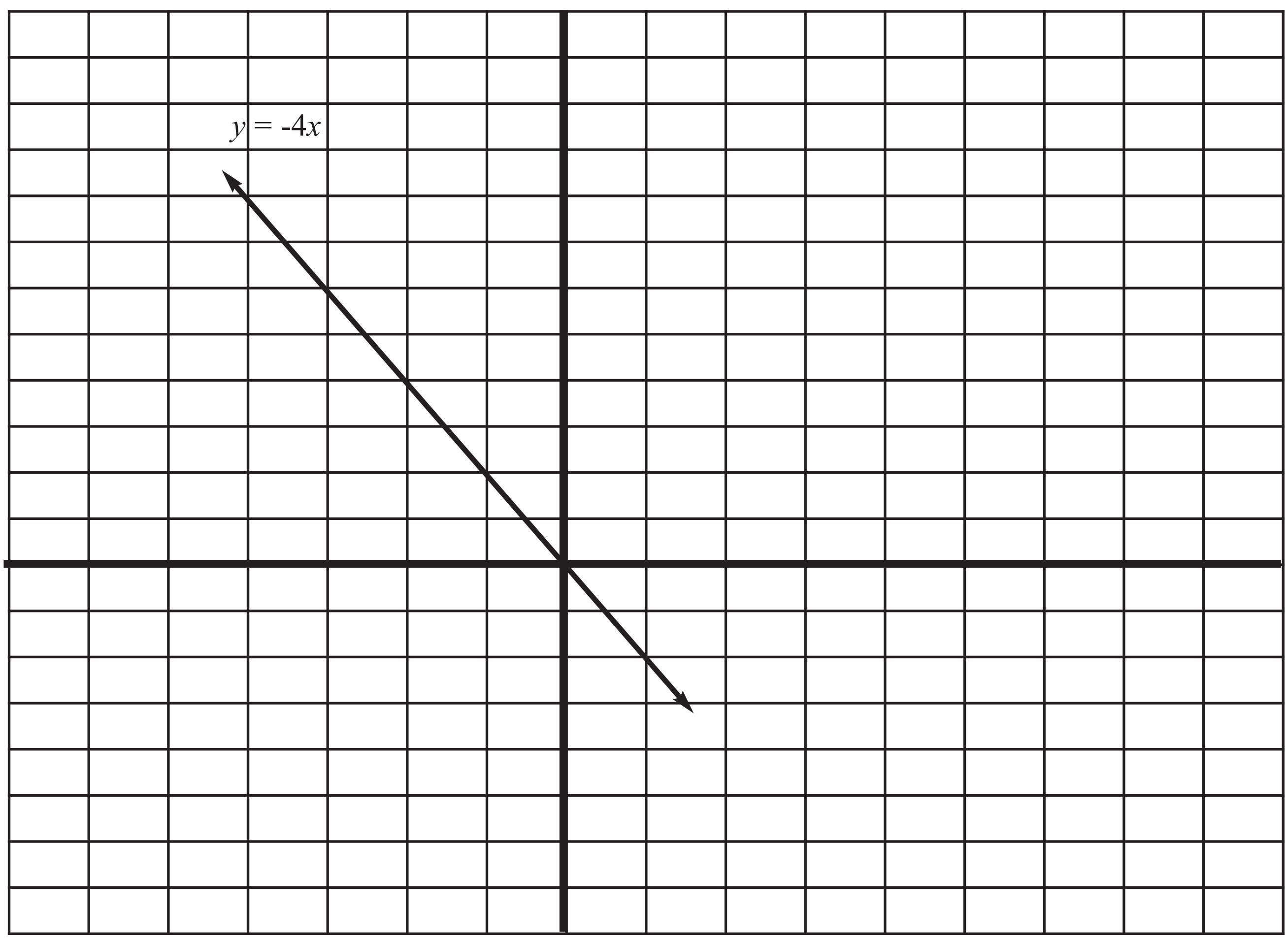
Part III:
One table and line have been completed. Complete the second table and line. Find the slopes of each line.

4.

$y = -2x$	
x	**y**
-4	8
-3	6
-2	4
-1	2
0	0
1	-2

5.

$y = -x$	
x	**y**
-4	
-3	
-2	
-1	
0	
1	



Slippery Slopes Answers

Part I: Complete each table and plot the points for each linear equation. Connect the points to make a line for each table.

1.

$y = x$	
x	**y**
-1	-1
0	0
1	1
2	2
4	4
6	6
10	10

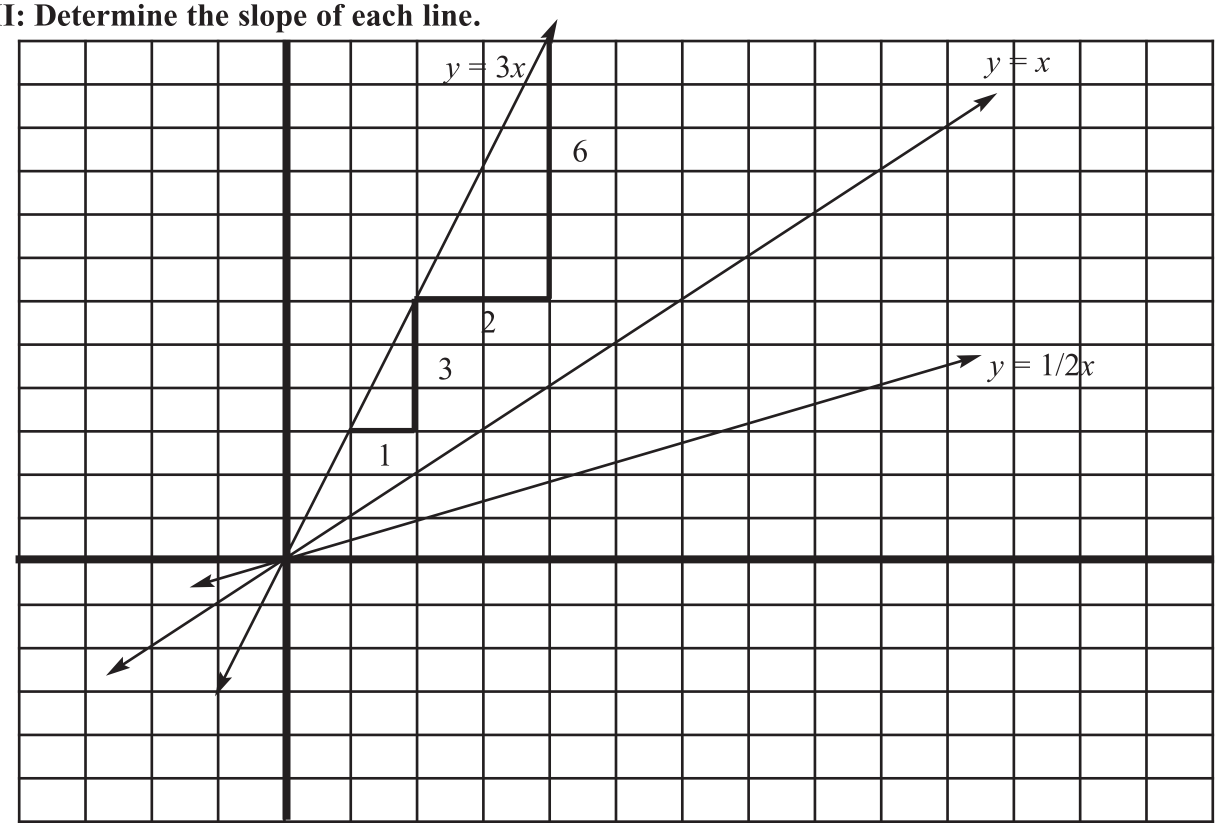
2.

$y = 3x$	
x	**y**
-1	-3
0	0
1	3
2	6
3	9
4	12

3.

$y = 1/2x$	
x	**y**
-1	-1/2
0	0
2	1
4	2
6	3
8	4
10	5

Part II: Determine the slope of each line.



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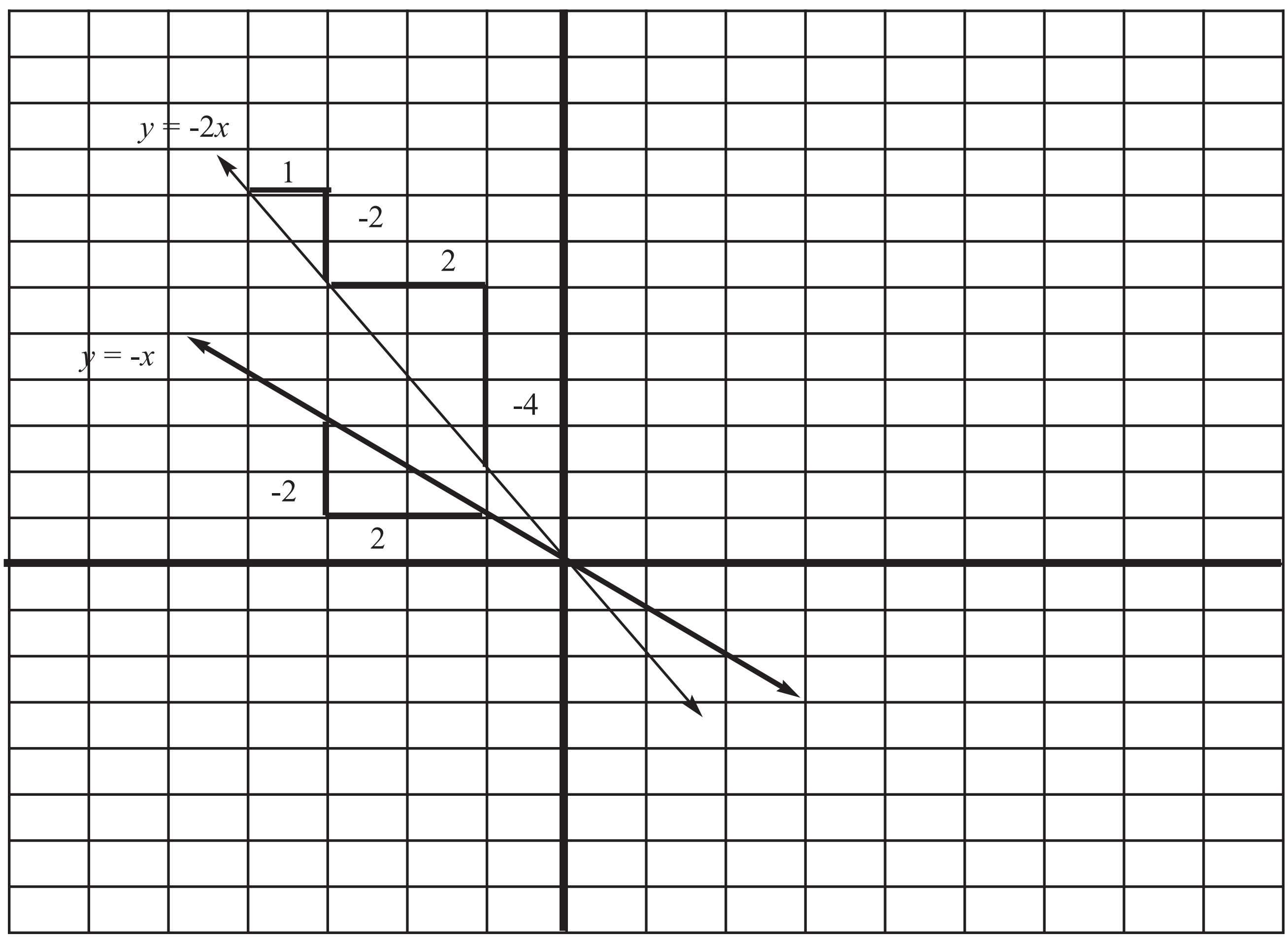
0	0
1	-2

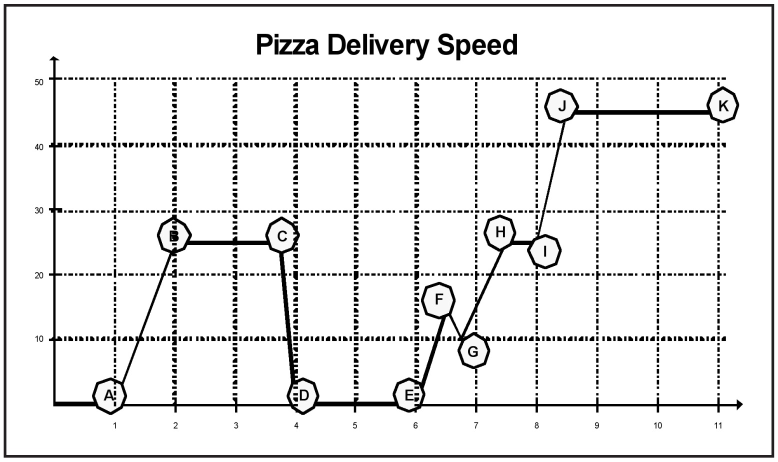
5.

y = -*x*	
x	**y**
-4	4
-3	3
-2	2
-1	1
0	0
1	-1



Read All About It

Use your knowledge of slope to complete and translate these stories.



Part One

Study the graph above showing the activity of a pizza delivery car. Answer the following questions with values of time (t). (For example, AB includes all time between A and B).

1. For what values of time (t) does the delivery car increase its speed?

2. For what values of t did the pizza delivery car stop to deliver a pizza?

3. At what time does the delivery car achieve its highest speed?

4. For what values of t did the driver slow down to make a turn onto a street?

5. For what values of t does the car drive at a constant rate?

6. At what values of t might the driver have come to a quick stop as he almost passed the house he was looking for?

Read All About It

Part Two
Use centimeter graph paper to sketch a graph that represents the following description for a hike going up a mountain for 5 miles. Make sure to label the axes.

The trailhead will be considered the start of the 5-mile hike. The trail went up gradually for 2 miles. Between miles 2 and 3, the trail flattened out. For the next half mile, the trail went slightly downhill. For the next mile it increased gradually, becoming very steep for the last half mile to the top of the mountain.

Read All About It

Part Three
Write a scenario and draw its graph. Share your graph with a partner or the group to check for accuracy.

Centimeter Grid Paper

Module E
Instructor's Guide

Module E: Spatial Reasoning

A. Module Goals
Use the transparency **Module Goals (T1)** to review the goals of the module.

Module E: Spatial Reasoning
The paraeducator will:
1. Use concrete methods to determine the connections between perimeter and area
2. Develop perimeter and area formulas for basic geometric shapes
3. Explore perimeter and area concepts in relation to circles
4. Use coordinate geometry to explain basic transformations

The concept of developing spatial reasoning is often overlooked in the mathematics curriculum. It is often assumed that students will understand spatial concepts by using formulas. Spatial reasoning is heavily linked to geometry skills. Geometry is more than just naming angles and shapes. While the foundation is set in the early elementary grades for basic skills, the ability to reason spatially involves experiences with problem solving and pattern identification for spatial concepts.

Goal 1: Use concrete methods to determine perimeter and area.

1.1 Lecture: Perimeter and Area Review

Note to Instructor: Remind participants to take notes during lecture sessions.

In grades K-4, students explore the basic concepts of perimeter and area conceptually. While no formal equations have been developed, students should have been looking for shortcuts and patterns to aid with formula development.

As a review, use the transparency/handout **Perimeter and Area (T2/H1).**

Perimeter
Distance around a shape or region.

Examples: fencing, wallpaper border,

Area
Amount of space inside a shape or region.

Examples: sod, paint, carpet

As participants work with these skills without formulas, it is important that they recognize and understand that there can be more than one result. Formulas often mislead students into thinking that there is a single answer. While formulas are easier to teach, participants must remember that the concept is the foundation.

It is important that students have experience with concrete materials such as Geoboards and more abstract models such as graph paper. While calculation of these concepts is a key skill, reasoning about the results is a more important skill for problem solving.

1.2 Discussion: Working with Perimeter and Area Models

The paraeducator will use graph paper models to explore the connections between perimeter and area.

Materials:
- Transparency/handout **Dot Paper (T3/H2)** (2 sheets per participant)

1.2.1 Steps
- This discussion is a combination of a group activity and discussion. Give participants time to experiment throughout the discussion.
- Have participants draw a square using four dots. Determine the perimeter and area of that square as a reference for all other information in this goal.
  - ▲ P = 4 units (from dot to dot vertically or horizontally is one unit)
  - ▲ A = 1 square unit (square units are the unit for area)
- Use the following rules:
  - ▲ Only vertical and horizontal lines will be used
  - ▲ No diagonals at this point
  - ▲ Each square must have a side touching another square
- Give the following scenarios and ask participants to complete them. Debrief after each example. It is important that participants share their examples. Record class examples on the overhead so that all participants can check the answers (solutions will not be given here because answers will vary greatly).
  - ▲ A = 2 sq units; give the P
    - o Two solutions will come up here, 1 x 2 and 2 x 1
    - o P = 6 units no matter which orientation
  - ▲ A = 4 sq units; give the minimum and maximum perimeter
    - o There are many orientations following the above rules
    - o Min P = 8 units; Max P = 10 units
  - ▲ A = 5 sq units, P = 10 units
    - o Many orientations; have participants check each other
  - ▲ A = 5 sq units, P = 12 units
    - o Many orientations; participants check each other
- Ask participants to answer the following based on their drawing experiences:
  - ▲ What shapes tend to minimize perimeter? (squares or rectangles as fewer sides are exposed)

- What strategy can be used to maximize perimeter? (shapes that are nonrectangular; the more sides, the higher the perimeter)
- What is known about shapes that have odd area values? (the shapes cannot be rectangular unless they are one straight row due to the odd number)
- What strategy was used for the last two examples with identical areas? (to get a higher perimeter value, they had to be nonrectangular; the shape needed many sides)

• Ask participants the following. These are higher-level questions that will require participants to design an experiment to answer the questions. The work may be done in their journals.
  - Area of rectangular shapes can be linked back to what elementary concept? (multiplication tables; factors; multiply length times width)
  - If the area of a square is 144 sq units, what is its perimeter? (in a square, all sides are equal. This means that the dimensions are 12x12; each side is 12 units. The perimeter is 48 units)
  - If the perimeter of a square is 36 units, what is its area? (a square with a perimeter of 36 units must have sides that are all 9 units. The area must be 9x9 or 81 sq units)
  - What happens to the area of square when each side length is doubled? (The area is quadrupled or multiplied by 4. This is the result of squaring 2; each side is doubled, meaning it becomes 2 x 2 times or 4 times larger)
  - Can we find the perimeter of a circle? (yes – the outside edge just happens to be curved)
  - Why did we not use diagonals? (the diagonal of a square is longer than the sides)
    - o The common reaction here is that the diagonal and sides are the same length. Do not let participants believe this is true. Use a ruler to measure if necessary.

***Note to Instructor*:** This is the Pythagorean Theorem, which says that the sum of the squares of the two legs is equal to the square of the longer side (hypotenuse).

Goal 2: Develop perimeter and area formulas for basic geometric shapes.

2.1 Activity: Going the Distance With Perimeter

The paraeducator will develop connections among basic perimeter formulas.

Materials:

• Handout **Going the Distance With Perimeter (H3)**

2.1.1 Steps

- Share the following observations from the previous goal:
  - ▲ Finding the perimeter of a shape can be time-consuming, especially if the numbers are large
  - ▲ Only shapes that were easy to count were used; this is not the general case
  - ▲ No formulas were used; simple counting was sufficient
- Teachers tend to make the mistake of rushing to formulas too soon; formulas are useful only if students know the reason for their existence.
- Review the fact that perimeter measures the distance around a region or shape.
  - ▲ This can be simple if the lines are on a grid
  - ▲ There must be a way to manage shapes that do not fit a grid or real-life examples that cannot be placed on a grid
  - ▲ This is the point of formulas; they generalize the work to be used in any situation and make calculations more efficient
- Have participants complete Part I of the handout **Going the Distance with Perimeter (H3).**
  - ▲ For all of these, the perimeter can be found by simply adding the side lengths together
  - ▲ As the class goes over the answers, ask for any shortcuts they may have seen rather than simply adding.
    - o 1. 8 in. (2 sets of sides are same)
    - o 2. 8 in. (all 4 sides are equal)
    - o 3. 12 ft. (no apparent pattern)
    - o 4. 12 cm (2 sets of sides are same)
    - o 5. 13 ft. (no apparent pattern)
    - o 6. 12 cm (all 6 sides are the same)
- Ask participants to complete Part II of the handout **Going the Distance with Perimeter (H3)**. They may skip the circle formula at this time.

Note to Instructor: Participants may need a quick review of shape names if they have not completed the K-4 Academy or do not possess the prior knowledge.

- Debrief answers as part of the next discussion.

2.2 Discussion: Pattern with Formulas

The paraeducator will use patterns to connect 2-D geometric shapes and derive the area formula for common 2-D shapes.

Materials:

- Completed handout **Going the Distance with Perimeter (H3)**
- Transparency/handout **Dot Paper (T3/H2)**
- Transparency **Analyzing Area (T4)**
- Handout **Common Area Formulas (H4)**
- Multiple colored overhead pens
- Scissors

2.2.1 Steps

- As formulas are discussed, remind participants that formulas are simply rules.
- As a group, debrief the answers for Part II of the handout **Going the Distance with Perimeter (H3)**. Test the formulas on the handout **Area Formulas (H4)** with the data from the diagrams as the group agrees upon the patterns.
  - ▲ *Square*: implies multiplying the side value by 4 (matches the observation of four sides being equal)
    - ○ $P = 4s$
$P = 4(2) = 8in.$
  - ▲ *Rectangle*: implies combining like data or multiplying each repeated number by 2, then adding the result (matches the observation of repeated data); L and W stand for length and width
    - ○ $P = 2L + 2W$
$P = 2(3) + 2(1) = 8in.$
  - ▲ *Triangle:* says to add all sides together (matches the observation for no apparent pattern)
    - ○ $P = a + b + c$
$P = 4 + 3 + 5 = 12$
  - ▲ *Parallelogram:* implies combining like data or multiplying each repeated number by 2, then adding the result (matches the observation of repeated data); a and b stand for two different sides
    - ○ $P = 2a + 2b$
$P = 2(2) + 2(4) = 12$
  - ▲ *Trapezoid:* says to add all sides together (matches the observation for no apparent pattern)
    - ○ $P = a + b + c + d$
$P = 1 + 3 + 4 + 5 = 13ft.$

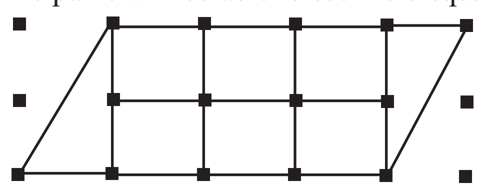
1. The rectangle formula is related to which other quadrilateral? Explain.
   (it is related to the parallelogram. A parallelogram is a "stepped on" rectangle. The formulas could use the same variables, L and W or *a* and *b*)

2. Can the perimeter of a square be determined with any other formula? Explain.
   (yes – it can be treated as a rectangle or parallelogram)

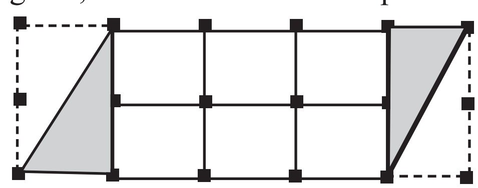
3. What is a general plan for perimeter if no formula is available?
   (the general plan is to just add the sides together as in the triangle and trapezoid)

4. What general formula could be created for a regular polygon (all sides equal) such as #6 in Part I? Explain.
   (the square is actually an example of a regular polygon as all of its sides are equal. A general formula could be [number of sides] x side measurement)

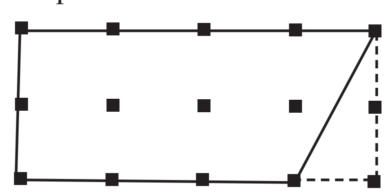
- The same type of analysis can be performed with area. Discuss the following points about area.
  - ▲ Easy to calculate if counting squares in a region
  - ▲ Difficult to calculate for irregular or curved regions
  - ▲ Calculations would be off if the area involved estimating for irregular or curved regions (to be covered later in this goal).
- Complete the activity using the transparency **Analyzing Area (T4)** and handout **Dot Paper (H2)** as a group and discuss the results throughout.
- Put up the transparency **Analyzing Area (T4)** and ask the participants to copy the top figure to their dot paper.
  - ▲ Ask participants to name the quadrilateral. (parallelogram)
  - ▲ Ask for the area of this quadrilateral. (8 sq units)
    - o Participants will be able to see the 6 square units – trace those out



    - o Where there is a diagonal, remind participants that we do not want to know how long the diagonal is, only what area is inside the shape
    - o In a different-colored pen, trace out the rectangles that contain the diagonal, which should be two squares each

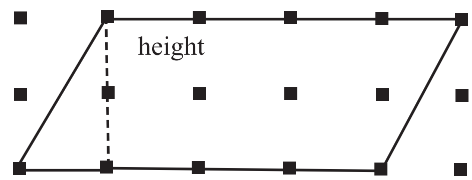


    - o Ask the participants what amount of area is shaded of the little rectangle. (1/2 of the region so the area shaded is 1 sq unit for one little rectangle)
    - o As that is true for each rectangle, the total area is 2 sq units
    - o Added with the 6 whole squares, the area is 8 sq units
  - ▲ Ask participants to cut off one of the triangles and match it to the other end

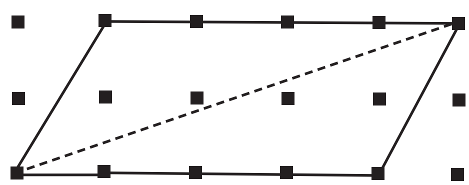


  - ▲ Ask what the new quadrilateral is. (rectangle)
  - ▲ Ask what its area is. (8 sq units; find by counting or from the previous exercise)
  - ▲ Ask for explanations of the relationship between the area of a parallelogram and the area of the rectangle. (they are the same)

- ▲ Besides counting, ask for a method of finding the area of a rectangle. (length x width)
- ▲ Ask for a way to find the area of a parallelogram. (should be similar to the rectangle because they have equal areas but the width is confusing)
  - o Remind participants that the numbers used for the area of rectangle would be 4 x 2
  - o The 4 is obvious from the length of the top and bottom
  - o Explain that the 2 is the *height,* which must be drawn vertically from the highest point to a lowest point

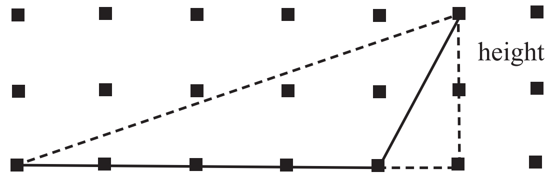


- Look at the formulas for rectangle and parallelogram on the handout **Common Area Formulas (H4).**
  - ▲ Ask if the rectangle and parallelogram could share a formula. (yes, if the rectangle was given as $A = bh$, then both would fit under the same formula)
- Restart the discussion with the original parallelogram. Have participants draw a new one on the handout **Dot Paper (H2)**.
  - ▲ Ask participants to draw a line from one corner of the parallelogram to the other.



- ▲ Ask what shapes have been created. (two equal triangles)
- ▲ Ask for the area of one triangle. (4 sq units)
  - o This is very difficult to count by hand because the lines are not perfect
  - o The parallelogram has an area of 8 sq units, and each triangle is half of that area, or 4 sq units
- ▲ Ask what formula could be used to calculate the area of any triangle. (A = 1/2 parallelogram = 1/2*bh*)
  - o Participants should be able to identify that the area of the triangle is half that of the parallelogram so the formula should be half of the parallelogram formula
  - o A key point to discuss is the *height* concept
  - o Use the formula created, A = 1/2 (4)
  - o As half of 4 is 2, height must be 2 for an area of 4 sq units
  - o Remind participants that *height* must go from the highest point to the lowest point with a vertical line

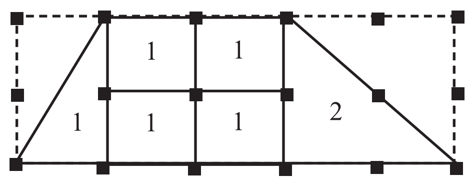
- Sketch this to show where to draw the *height*



- Participants will question how the *height* can be outside of the figure – it does not matter where the line is drawn as it is in reference to the shape

▲ Check the group formula against the handout **Area Formulas (H4)**. It should match the group findings

- For the final discussion point, start out with a trapezoid on the bottom of the transparency **Analyzing Area (T4).** Have participants copy it to their handout **Dot Paper (H2)**.
  - ▲ Ask participants for the name of the initial shape. (trapezoid)
  - ▲ Ask participants to calculate the area of the region. (7 sq units)
    - Similar methods for area must be used for counting as were used with the initial parallelogram
    - Sketch out squares to assist participants in seeing 7 sq units



  - ▲ Ask participants to duplicate the trapezoid and cut both out.
  - ▲ Rotate one so that it fits into the other trapezoid, creating a familiar shape. (it should be a parallelogram)
  - ▲ Ask for the relationship between the area of the single trapezoid and the new quadrilateral. (the area of the trapezoid is half the area of the parallelogram)
  - ▲ Ask for the relationship between the measure of the base of the parallelogram and the measure of the trapezoid bases. (the two bases of the trapezoids add together to make the base of the new parallelogram)
    - This will be difficult for many participants to see because of the rotated piece
    - The new base of the parallelogram is the sum of the two different bases for the trapezoid
  - ▲ Ask what formula could be written to find the area of any trapezoid. ($A = 1/2(b_1 + b_2)h$)
    - Participants will need help working through this concept
    - A common error is to end up with the triangle formula
    - The trapezoid formula says that there is a relationship to the parallelogram but notes that trapezoids have different bases

- The square was not included in this exercise but needs to be addressed.
  - ▲ Ask participants if any formula they have learned could be used instead of the one given on the formula sheet. (yes – as learned from perimeter, squares are just a type of rectangle or parallelogram)
  - ▲ Ask why the formula reads $A = s^2$. (shows that the sides are the same)
    - o Participants may need help remembering exponent rules
    - o The regular formula would read $A = bh$. If the sides are the same, raising the number to the second power is appropriate
    - o The formula could have easily been stated as $A = b^2$ or $A = h^2$
- Conclude the discussion stressing the fact that most area computation could have been done without using formulas. Formulas just made the work more efficient.
- Emphasize that, in reality, only one single formula needs to be memorized, $A = bh$. When students feel overwhelmed by the number of formulas to be memorized, they have not learned the appropriate base knowledge.

Goal 3: Explore perimeter and area concepts in relation to circles.

3.1 Lecture: Perimeter and Circles

***Note to Instructor**:* Remind participants to take notes during lecture sessions.

Up to this point, only polygons have been considered when dealing with area and perimeter. Another major skill introduced in the middle grades is working with circles. Circles present difficulty because of the curved line. Curved lines make estimation difficult when circles are placed on grids.

It is possible to calculate the perimeter of a circle. The perimeter of a circle is called *circumference.* Remind participants to add this to the handout **Going the Distance with Perimeter (H3).**

Using a string or a piece of yarn is an easy way to measure circumference. While not perfectly accurate, it allows for the curvature to be measured. Participants will get practice measuring the perimeter in the following activity.

To be able to use the appropriate formulas, the concept of pi must be discovered. Do not share with participants that this is the goal of the following activity.

3.2 Activity: Going in Circles

The paraeducator will define pi through concrete experiences.

Materials:
- Transparency/handout **Going in Circles (T5/H5)**
- Yarn or string
- Circular objects to trace or geometry compasses
- Calculators
- Rulers
- Scrap paper with a blank side

3.2.1 Steps
- Ask each participant to draw or trace at least three circles of different sizes on their paper.
- For each circle, have participants find the circumference using the string (may need to show what this looks like).
  - ▲ Lay the string around the circle
  - ▲ Straighten the string and measure that distance on the ruler
  - ▲ It does not matter whether standard or metric units are used
- For each circle, have participants measure the *diameter*.
  - ▲ The *diameter* is the distance across the circle going through the center.
- For each circle, calculate the ratio of *circumference to diameter*.
  - ▲ Participants may need to be reminded how to calculate a ratio
    - o The ratio of *10 to 2* is 10/2 = 5
  - ▲ It is necessary to use calculators due to the time required of decimal division calculation
- Walk around the room to make sure that calculations are being performed correctly.
  - ▲ It is a common error to put them in the calculator backward
- Have participants share the results.
  - ▲ All answers should be close to 3.14 – an approximation of the value of pi (π)

3.3 Lecture: Formulas and Circles

***Note to Instructor**:* Remind participants to take notes during lecture sessions.

The concept of pi has many links to higher-level mathematics concepts. Too many students use the key on their calculator and have no idea that the symbol is actually a number. Pi is a unique number in that it is a nonterminating and nonrepeating decimal value. The number 3.14 is a drastic approximation but is commonly used in middle school texts.

The formulas listed for both *circumference* (perimeter) and *area* involve the concept of pi. Pi is important for the situations where string or grids can not be used such as measuring around a large silo or even measuring around the earth.

One other important concept listed in the formulas is the concept of *radius*. A *radius* starts at the center of a circle and goes to the outside of the circle like a bicycle spoke. All *radii* (plural for radius) are the same length by definition of a circle. According to the technical definition of a circle, all points are equidistant from a central point. While this definition may be too technical for middle school students, it gives some perspective to the concept of a *radius.* Sketching a diagram showing diameter and radius would be helpful so that participants understand the terms and that the radius is simply half of the diameter.

Show the following diagram, transparency/handout **Going in Circles (T5/H5).** Calculate the perimeter with a string and with the formula. Calculate the area by counting and estimating and using the formula. This should confirm the use of pi (use 3.14) and the formulas. The dot paper containing the circle has been resized so that the dots are 1cm apart. Have participants use the metric side of their ruler when measuring.

The perimeter using the formula should be

$$C = 2\pi r = 2(3.14)(4) \approx 25.12cm$$

$$C = \pi d = (3.14)(8) \approx 25.12cm$$

The area using the formula should be should be

$$A = 2\pi r^2 = (3.14)(4)^2 =$$

$$= (3.14)(4)^2 \approx 50.24cm^2$$

****Note to Instructor**:* Extra information is provided below. Share if time allows for those who are interested.

While this goal is about circles, this also seems like an appropriate place to mention volume and surface area. The concepts of perimeter and area work for two-dimensional shapes. Volume and surface areas apply to three-dimensional shapes such as cubes and cylinders. Volume and surface are concepts dealt with in the middle school grades.

Participants who learn perimeter and area through hands-on experiences should be able to transition to volume and surface area with ease. While there are many new formulas to learn, participants should be able to recognize similarities.

If available, bring stackable washers or paper coasters as an example of the easy transition to volume. A cylinder (like an orange juice can) is simply a stack of circles. As volume adds a third dimension, height is the added dimension for a stack of circles. The idea is that once the area of one circle in known, the volume can be found simply by multiplying the number of circles involved. No new formula is necessary to find the volume of the cylinder.

Comparing the mathematical equations, $A = \pi r^2$ (for area of one circle) and $V = \pi r^2 h$, it is obvious that the formulas are directly related.

This kind of spatial reasoning works for the cube (as a stack of squares) or a prism (the stack depends on the type of base). Formulas for the cone are similar but require additional explanation of the related volumes. This is beyond the scope of this module.

Participants need to realize that geometry builds on itself. Unless they put forth effort to understand related basic concepts, participants will be overwhelmed by the number of definitions required to survive geometry.

Goal 4: Use coordinate geometry to explain basic transformations.

4.1 Lecture: Understanding Transformations

Note to *Instructor: Remind participants to take notes during lecture sessions. Sketches are important to this lecture.

Materials:
- Transparency/handout **Transformations (T6/H6)**
- Pattern blocks
- Blank overhead transparencies
- Overhead pens

A common geometry skill involves *transformations* (see the transparency/handout **Transformations [T6/H6]**).

> ***Transformation***
> Mathematically, a *transformation* is a one-to-one correspondence between points in a plane.

This means that if a triangle had points A, B, and C, its *transformation* must have the same identifiable points, called A', B', C' (said A prime, B prime, and C prime), even though they may have moved. That is, the same basic shape is maintained but location or size has been transformed.

Several different types of basic transformations are learned in early elementary grades. These basic concepts continue through upper-level mathematics, especially in high school and college algebra.

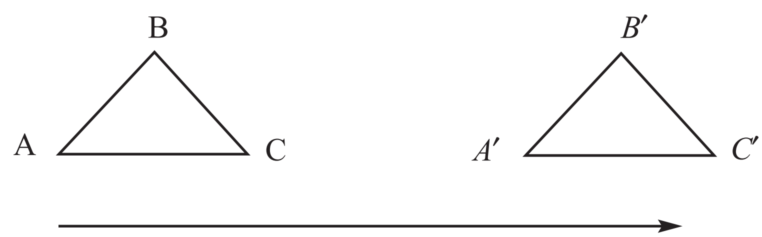
For this goal, three transformations will be covered: translation, reflection, and size.

The *translation* is the most elementary of the transformations.

Translation
A motion of the figure that moves every point of the figure a specified distance in a specified direction along a straight line.

An example of a translation is a child sliding down a slide. The child remains the same, but has moved locations along a line that was the slide. Often, *translations* are called *slides* in early elementary grades. *Translations* keep the figures *congruent* or equal.

To demonstrate on the overhead, use a pattern block and trace its image. Label its points, A, B, C, and so on. Draw a line of translation with an arrow showing the direction of the translation. Physically slide the shape along the line and retrace its image. Label those points *A′*, *B′*, *C′*, etc., paying special attention to the placement of the labels, which should be identical to the original image.



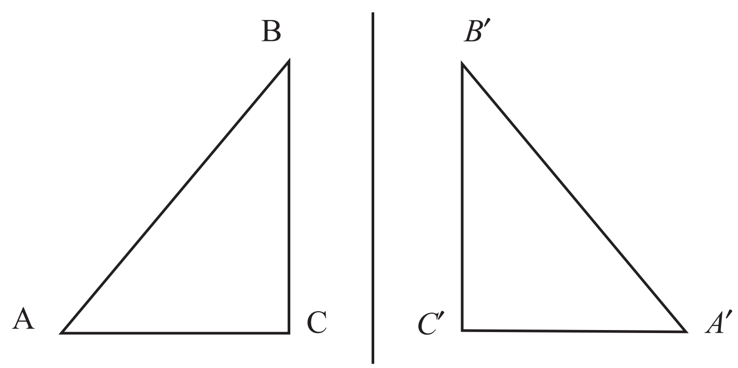
A second type of transformation is a *reflection*.

Reflection
"Flips" an image about a line of reflection.

A *reflection* is commonly called a *flip* in early elementary grades. It takes each point and places it across some line of reflection in the same distance from that line.

A common example is to think of a mirror reflection. The figures reflected remain the same, or *congruent*, but the image has been flipped.

To demonstrate on the overhead, draw a shape or trace a pattern block and label the points. Draw a line of reflection. Flip the image, being careful to label points correctly so that the transformation represents a reflection and not simply a translation. Some reflections are only identifiable if their points are labeled because they are symmetric figures.



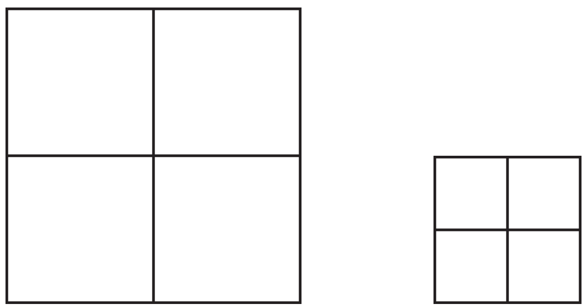
The last type of transformation covered here is a *size* transformation.

> ***Size transformation***
> Reduces or increases a figure by a specified scale factor or ratio.

Size transformations utilize *similar* figures rather than congruent figures because the figure changes size but maintains its basic properties.

A common use of *size transformations* is in perspective drawing.

To demonstrate *size transformations*, draw or trace a shape on the overhead large enough to easily reduce it in size. A *scale factor* of 2 says that the resulting similar figure is twice as big as the original. A *scale factor* of one half says that the resulting similar figure is half as big as the original.



This figure started out as a 2 x 2 figure. Using a scale factor of one half, all dimensions must be reduced by one half. The side length of the original square was 2 units, so the new square must have a side length of 1 unit. The figures are similar, but the size has been transformed.

It is important to note that the reduction in area for the above figure does not correspond to a ratio of one half. The area of the original was 4 sq units. The area of the final figure is 1 sq unit. (The effect of scale factor on area is beyond the scope of this Academy.)

While it is important to identify types of transformations, the link to coordinate geometry is the key spatial reasoning component. This will be explored in the following activity.

4.2 Activity: Move It

The paraeducator will use coordinate geometry skills to develop translation, reflection, and size transformations.

Materials:
- Transparency/handout **Move It (T7/H7)**
- Transparency **Move It – Answers (T8)** (optional)
- Straightedge or ruler

4.2.1 Steps

- In pairs, have participants complete the handout **Move It (H7).**
- Use the transparency **Move It (T7)** to show the answers for the graphs and set questions (may also use the transparency **Move It – Answers [T8]**). Make sure to show coordinates on the graph either with the ordered pair or with the A, A'concept.
  - ▲ Set 1:
    - o Note that for each new diagram a prime symbol was added to keep track of associated points
    - o Participants need to see the use of coordinate patterns to predict the new coordinates depending on the movement on the x or y axis
  - ▲ Set 2:
    - o It is important that participants paid special attention to the orientation of the original figure and its distance from the axes to get proper placement for the reflection
  - ▲ Set 3:
    - o Remind participants that the triangles are similar and not congruent
    - o Make sure participants are carefully reading the diagram and the direction of reference on the scale factor questions.

Note to Instructor: Make sure that participants realize that when doing transformations, shapes may overlap and merge. This is the case on the final exam.

5.1 Final Assessment

Paraeducators will use their notes and handouts to assist them in completing the final assessment for Algebraic Concepts and Spatial Reasoning Academy.

Use the handout **Final Assessment, Algebraic Concepts and Spatial Reasoning Academy (H8).** Allow 60 minutes for participants to complete the assessment. To assist in grading the assessment, answers to the questions are provided under each question. **Please ensure that the answers are not released to the students before they complete the assignment.**

Instructor's Grading Key for the Final Assessment

Grading is recorded and based upon an individual total of 100 possible points assigned as follows:

Module B: (20 points)

1. For the following sample space of beads in a sack, answer the questions. (5 points)
   S = {*green, yellow, blue, red, white*}

   a. What is the probability of choosing a blue bead on a single draw?
   Answer: P(blue) - 1/5

b. What is the probability of choosing a yellow <u>or</u> green bead on a single draw?

Answer: $P(yellow \cup green) = 2/5$

Note: These are mutually exclusive so the individual probabilities may be added.

c. What is the probability of choosing a purple bead on a single draw?

Answer: $P(purple) = 0$

d. Are these beads mutually exclusive? Explain.

Answer: Yes, once a color is drawn on a single draw, no other color can occur.

2. Evaluate the following without a calculator. (5 points)

a. (-2) + (-4) = -6

b. 2 – 7 = -5

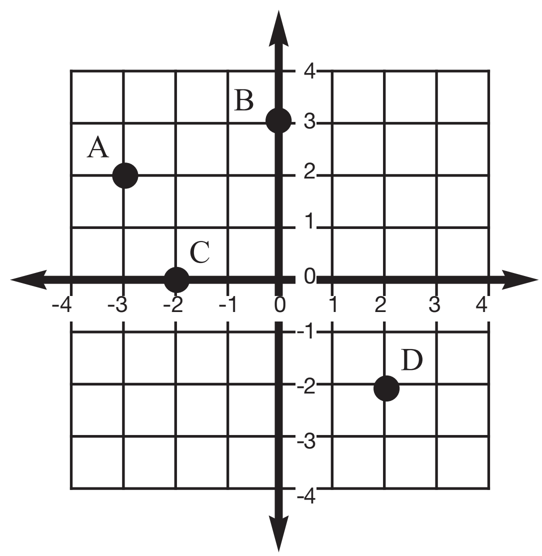
c. 6 – (-8) = 14

d. 20/-2 = -10

e. (-3) • 8 = -24

3. Name the following coordinates. (4 points)

Answer:
A (-3, 2)
B (0, 3)
C (-2, 0)
D (2, -2)



4. Which aspect of a coordinate would need to change to move a point from quadrant I to quadrant 2? (3 points)

Answer: The x coordinate would need to change in sign from positive to negative.

5. Which aspect of a coordinate would need to change to move a point form quadrant 3 to quadrant 1? (3 points)

Answer: The x and y coordinate would need to change from both negatives to both positives.

Module C: (20 points)

1. Complete the next three terms: (2 points)

   3, 6, 12, 24, 48, __**96**__, __**192**___, ___**384**___

2. Describe the pattern you used from #1. (2 points)

   Answer: The pattern seems to double every number to get the next term.

3. Complete the following table for the given function. Show your work for the substitutions. (5 points)

y = 2x - 1	
x	**y**
-1	-3
0	-1
1	1
2	3
4	7
6	11
10	19

4. Translate and solve the following. Show all steps in solving the equations. (4 points each)

   a. 9 plus a number p is 17

   Answer: $9 + p = 17$

   $$\begin{array}{rcl} \cancel{9} + p &=& 17 \\ \underline{-\cancel{9}} & & \underline{-9} \\ p &=& 8 \end{array}$$

   b. Negative 4 times a number w is 20.

   Answer: $-4w = 20$

   $$\frac{\cancel{-4}w}{\cancel{-4}} = \frac{20}{-4}$$

   $$w = -5$$

5. From the following table, give the algebraic rule to predict the *nth term.* Give the result for the *100th term* using your rule. (3 points)

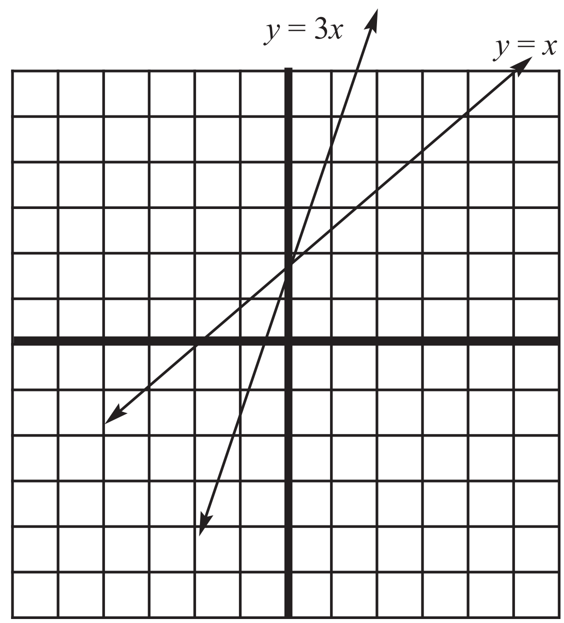
Term Number (n)	1	2	3	4	5	6	7	*n*	…100
Sequence	4	8	12	16	20	24	28	***4n***	**400**

Answer: The *nth term* is 4*n.*

This pattern is dependent on the term number. While it looks as if the *nth term* is n+4, plugging in the term number as *n* does not produce the required sequence. The difference becomes the coefficient or part of the product. No additional addition is required to produce the sequence.

Module D: (30 points)

1. Which of the following lines has a greater slope? Explain. (4 points)



Answer: $y = 3x$ has the greater slope
It is steeper than the other line, meaning it increases faster upward as it moves right than the other line.

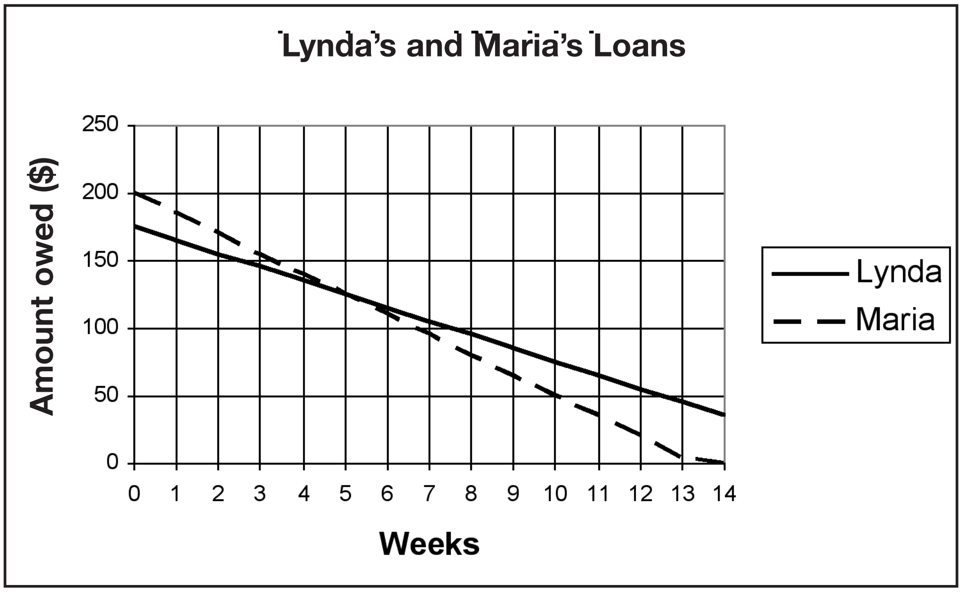
2. Determine the slope of each line using any method you wish. Verify your answer using any method. (6 points)

Answer: $y = 3x$ slope: 3
$y = x$ slope: 1

Possible methods include:
- Slope is found as the coefficient. It may be read from the equation
- Use points to verify slope using triangle and calculate the ratio of $\frac{rise}{run}$

3. Use the following chart to answer the questions.

Lynda and Maria take out a loan from their parents. Maria borrowed \$200 and pays back \$15 a week. Lynda borrowed \$175 and pays back \$10 a week.



a. Are these lines increasing or decreasing? (5 points)
Does this make sense with the data? Explain.

Answer: These lines are decreasing. This makes sense with the data as the amount of money each owes is decreasing by the weeks.

b. At what week do they owe the same amount? What is the amount? (5 points)

Answer: At week 5 they own \$125 each.

c. Who reaches zero first? (4 points)

Answer: Maria.

d. How is your answer to c) above affected by the slope of the lines? (6 points)

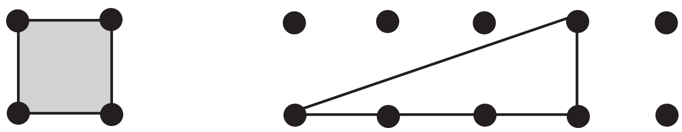
Answer: The slope of Maria's line must be larger because it is a steeper decrease. This means she would get to zero more quickly as the rate would be increased.

Module E: (30 points)

1. Draw a figure on the centimeter graph paper that has an *area* of 6 sq units and a *perimeter* of 14 units. Use only squares, no diagonals. (4 points)

    Answer: There are many solutions to this problem. Make sure the drawing has 6 squares and 14 as the perimeter.

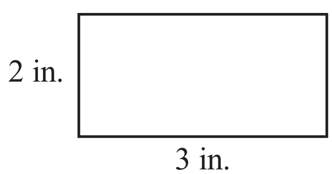
2. Find the *area* of the following triangle in *square units.* Remember that one square (4 dots) is the base unit. (4 points)



    Answer: The area is 1.5 or 1-1/2 square units.
    The rectangle it sits in has an area of 3 sq units. The triangle is half of that area or 1.5 sq units.

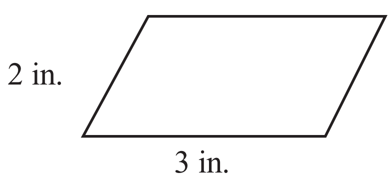
3. *Draw* each scenario and *calculate* the *perimeter.* Use your formulas from the handout where necessary. Drawings do not have to be to scale. Make sure to label. Use 3.14 for π where appropriate.

    a. Rectangle with a length of 3 inches and a width of 2 inches. (4 points)



$P = 2L + 2W$
$P = 2(3) + 2(2)$
$P = 6 + 4$
$P = 10$ *in.*

    b. Parallelogram with sides of 3 inches and 2 inches. (4 points)

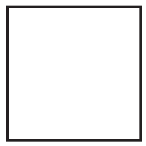


$P = 2a + 2b$
$P = 2(3) + 2(2)$
$P = 6 + 4$
$P = 10$ *in.*

4. *Draw* each scenario and *calculate* the *area.* Use your formulas from the handout where necessary. Drawings do not have to be to scale. Make sure to label. Use 3.14 for π where appropriate.

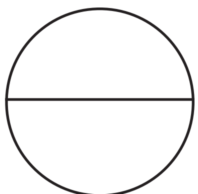
   a. Square with sides of 2 ft. (4 points)

2 ft.



$A = s^2$
$A = (2)^2$
$A = 4 \text{ ft}^2$

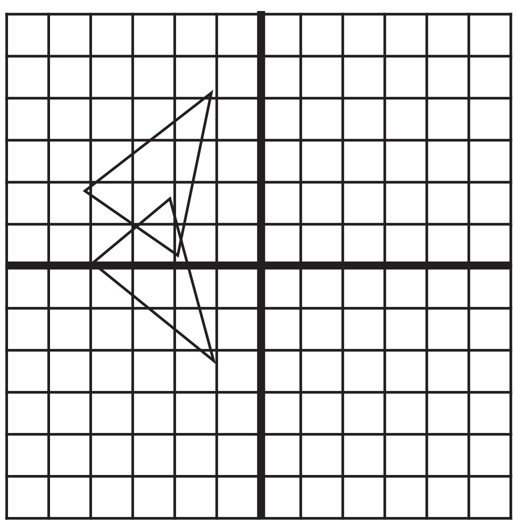
   b. Circle with a diameter of 4 cm. (4 points)



$A = \pi r^2$
$A = (3.14)(2)^2$
$A \approx 12.56\ cm$

A diameter of 4 means a radius of 2 cm.

5. *Reflect* the following figure across the *x axis.* Make sure to list the original coordinates and the new coordinates. (6 points)



Answer:

Coordinates of the original: A (-2, -1)
B (-4, 1)
C (-1, 4)

Coordinates of the reflection over the x axis: A'(-2, 1)
B'(-4, -1)
C'(-1, -4)

Module E
Handouts

Perimeter and Area

Perimeter

Distance around a shape or region.

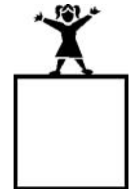
Examples: fencing, wallpaper border

Area

Amount of space inside a shape or region.

Examples: sod, paint, carpet

Dot Paper

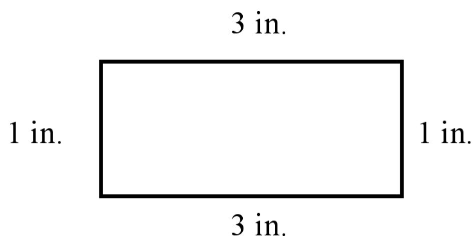


Going the Distance With Perimeter

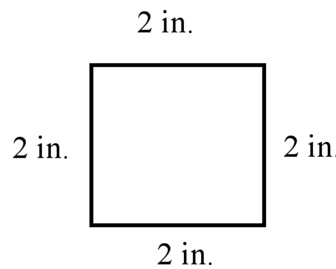
Find the perimeter of each of the following. Make use of the definition of perimeter and look for patterns in the processes. Drawings are not necessarily drawn to scale.

Part I:

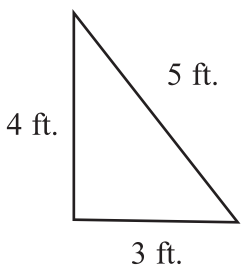
1.



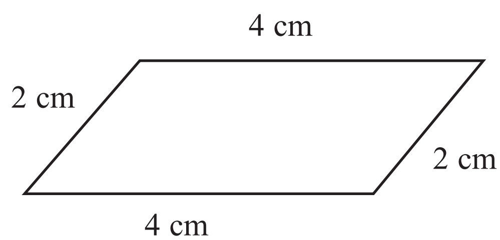
2.



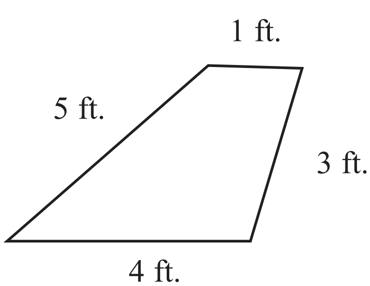
3.



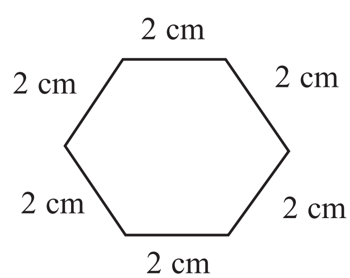
4.



5.



6.



Part II:

Use the patterns noted in Part I to analyze the common perimeter formulas. Describe how each formula works below.

Square → $P = 4s$
Rectangle → $P = 2L + 2W$
Parallelogram → $P = 2a + 2b$
Triangle → $P = a + b + c$
Trapezoid → $P = a + b + c + d$
Circle → $C = 2\pi r$ or $C = \pi d$

Answer the following questions based on your explanation above.

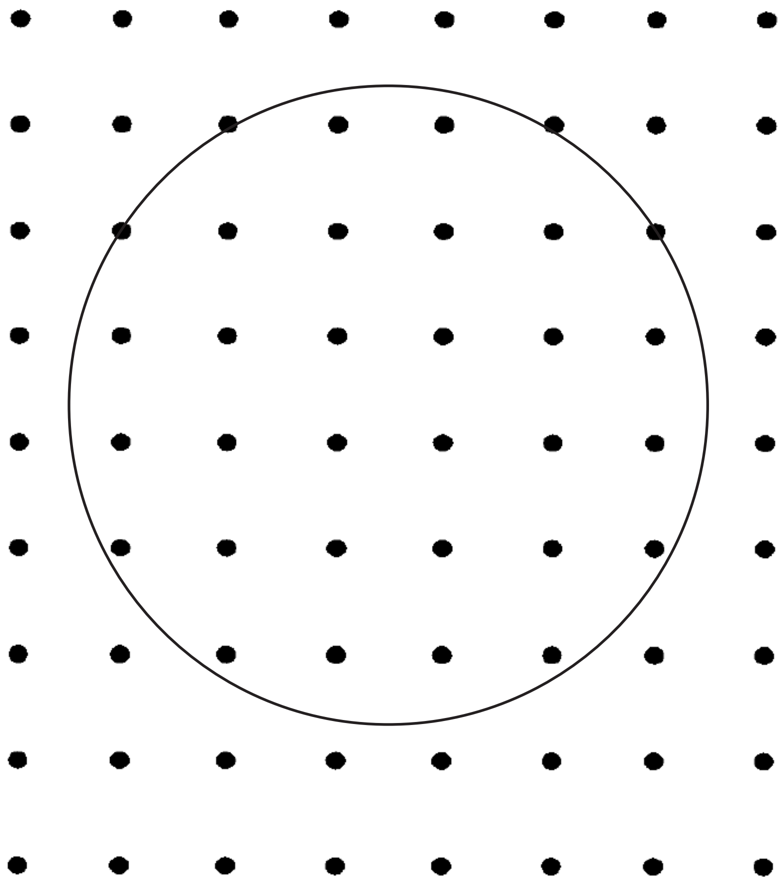
1. **The rectangle formula is related to which other shape? Explain.**

2. **Could the perimeter of a square be determined with any other formula? Explain.**

3. **What is a general plan for perimeter if no formula is available?**

4. **What general formula could be created for a regular polygon (all sides equal) such as #5 in Part I? Explain.**

Going the Distance With Perimeter



Common Area Formulas

Square → $A = s^2$

Rectangle → $A = lw$

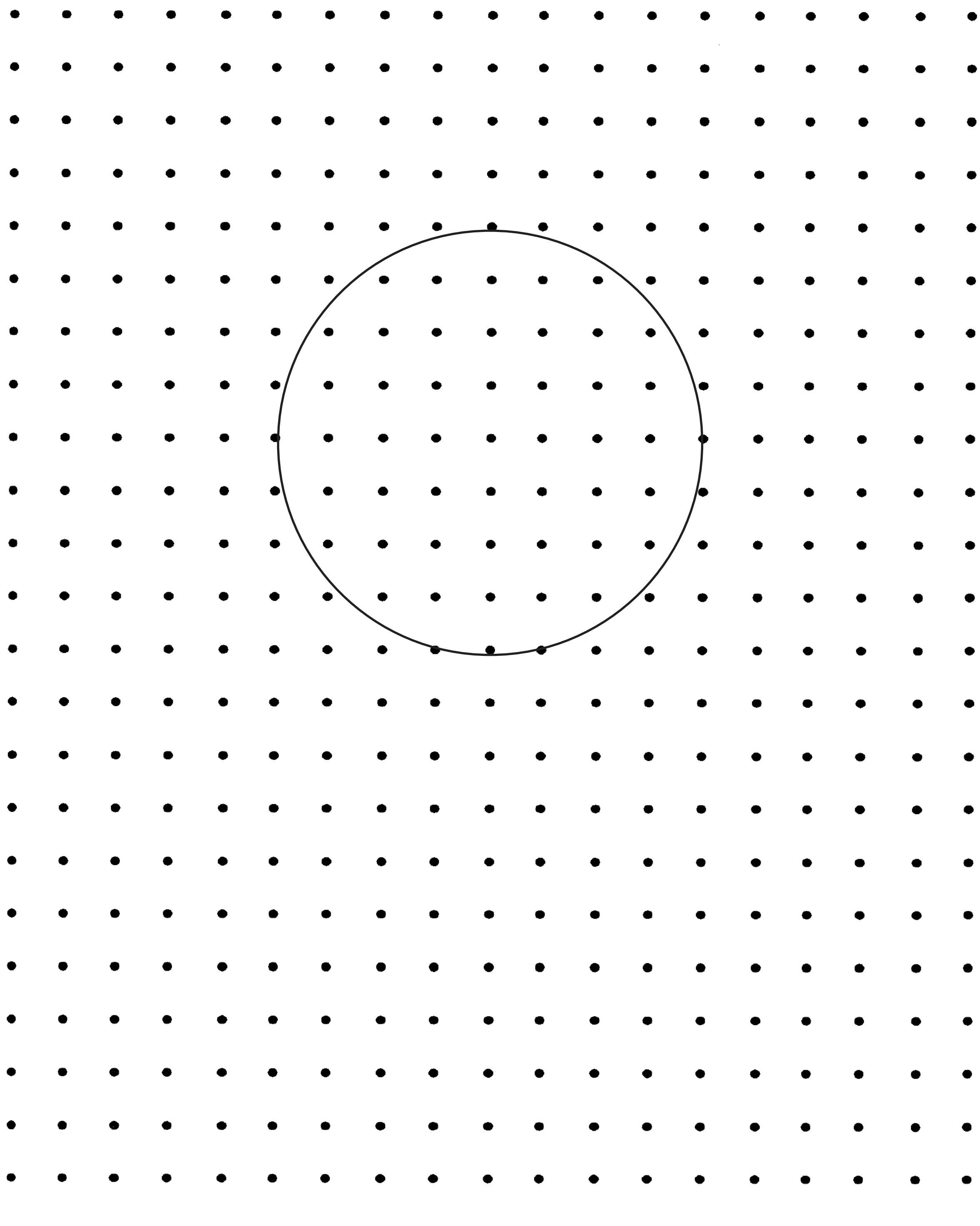
Parallelogram → $A = bh$

Triangle → $A = 1/2bh$

Trapezoid → $A = 1/2h(b_1 + b_2)$

Circle ? → $A = \pi r^2$

Going in Circles



Transformations

Transformation:

A one-to-one correspondence between points in a plane.

Types:

Translation:

A motion of the figure that moves every point of the figure a specified distance in a specified direction along a straight line.

- Figures must be congruent
- Orientation does not change
- Position changes

Reflection:

"Flips" an image about a line of reflection.

- Figures must be congruent
- Orientation and position changes

Size:

Reduces or increases a figure by a specified scale factor or ratio.

- Figures are *similar*
- Size changes by *scale factor* or ratio

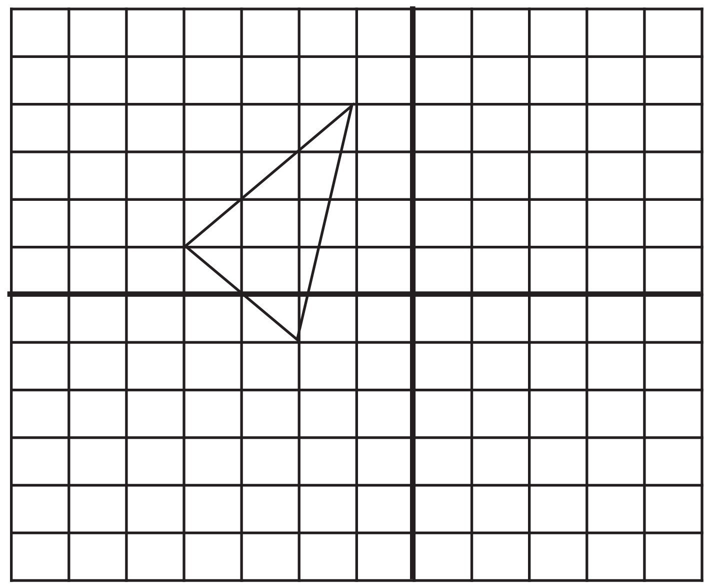
Scale Factor:

- A *scale factor* of 2 says that the resulting similar figure is twice as big as the original.
- A *scale factor* of one half says that the resulting similar figure is half as big as the original.

Move It

Complete each set of questions for the appropriate graph.

Set 1:

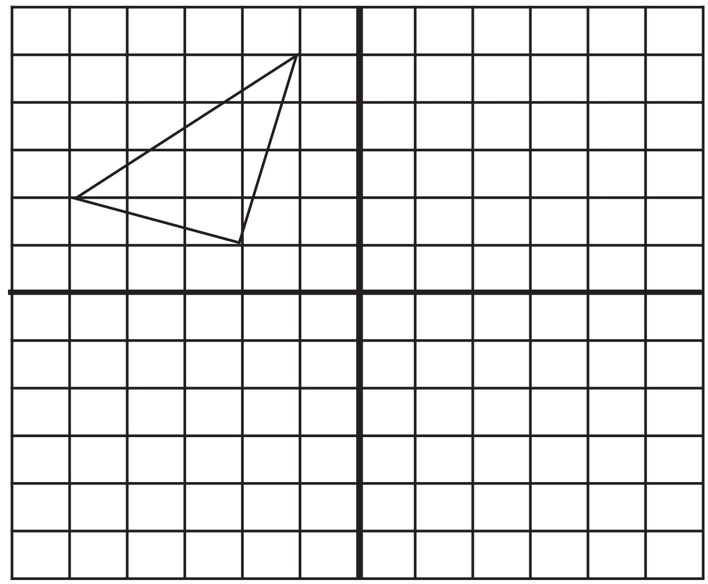


1. **Label the original coordinates for the associated points on the graph.**
2. **Translate the figure 5 units to the right. Draw the new figure.**
3. **Label the new points on the translated figure.**
4. **What pattern do you see in the coordinates for a translation of five units to the right?**

5 Which pattern would you use with the coordinates to translate the original figure 3 units left? Give your new expected coordinates and test your pattern above by counting.

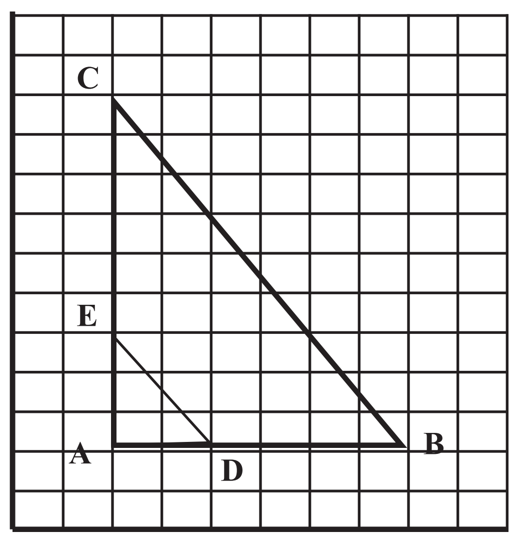
6. **Which pattern would you use with the coordinates to translate the original figure 4 units down? Give your new expected coordinates and test your pattern above by counting.**

4. **Which pattern do you see in the coordinates for a figure reflected over the y axis?**



5. **Which pattern would you use with the coordinates to reflect the original figure over the x axis? Give your new expected coordinates and test your pattern above by counting.**

Set 3:



Triangles ABC and ADE are similar.

1. **Find the ratio of $\frac{AB}{AD}$.**

2. **Find the ratio of $\frac{AC}{AE}$.**

3. **Determine the scale factor from *ADE* to *ABC*.**

4. **Determine the scale factor from to *ABC* to *ADE*.**

______________________ (Name)

______________________ (Date)

Final Assessment
Algebraic Concepts and Spatial Reasoning

<u>Using your notes and handouts</u> from the Academy *Algebraic Concepts and Spatial Reasoning*, complete the following assessment.
Time allowed: 60 minutes

Module B: (20 points)

1. For the following sample space of beads in a sack, answer the questions. (5 points)

   $S = \{green, yellow, blue, red, white\}$

   a. What is the probability of choosing a blue bead on a single draw?

   b. What is the probability of choosing a yellow <u>or</u> green bead on a single draw?

   c. What is the probability of choosing a purple bead on a single draw?

   d. Are these beads mutually exclusive? Explain.

2. Evaluate the following without a calculator. (5 points)

   a. (-2) + (-4)

   b. 2 – 7

   c. 6 – (-8)

   d. 20/-2

   e. (-3) • 8

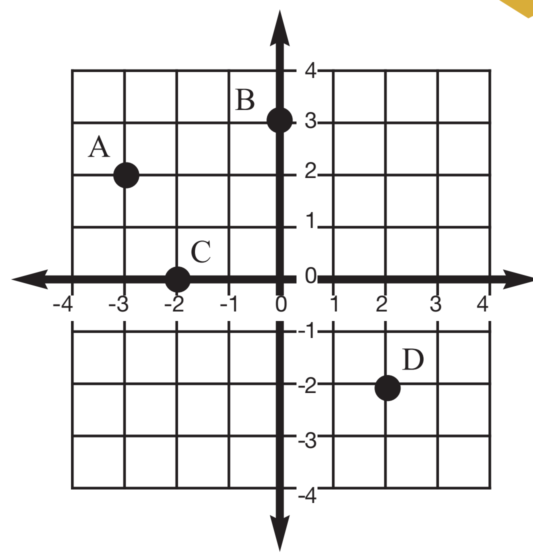
3. Name the following coordinates.
   (4 points)

   Answer:
   A =

   B =

   C =

   D =



4. Which aspect of a coordinate would need to change to move a point from quadrant I to quadrant 2? (3 points)

5. Which aspect of a coordinate would need to change to move a point form quadrant 3 to quadrant 1? (3 points)

Module C: (20 points)

1. Complete the next three terms: (2 points)

   3, 6, 12, 24, 48, ________ , ________, _________

2. Describe the pattern you used from #1. (2 points)

3. Complete the following table for the given function. Show your work for the substitutions. (5 points)

y = 2x - 1	
x	**y**
-1	
0	
1	
2	
4	
6	
10	

4. Translate and solve the following. Show all steps in solving the equations. (4 points each)

    a. 9 plus a number p is 17

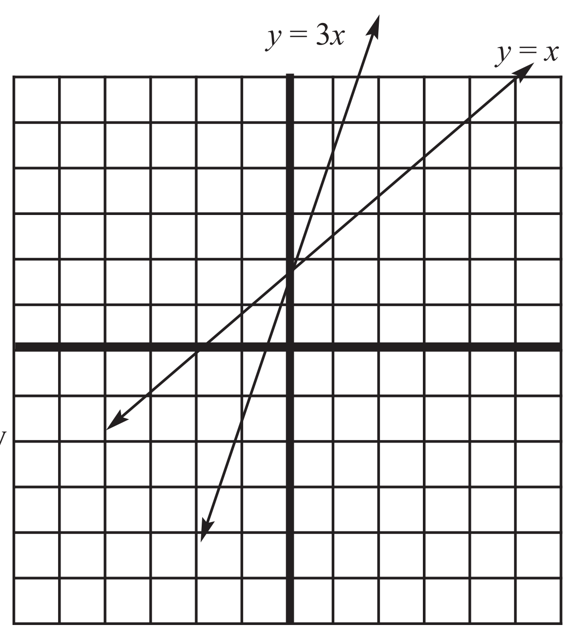
    b. Negative 4 times a number w is 20.

5. From the following table, give the algebraic rule to predict the *nth term.* Give the result for the *100th term* using your rule. (3 points)

Term Number (n)	1	2	3	4	5	6	7	*n*	…100
Sequence	4	8	12	16	20	24	28		

Module D: (30 points)

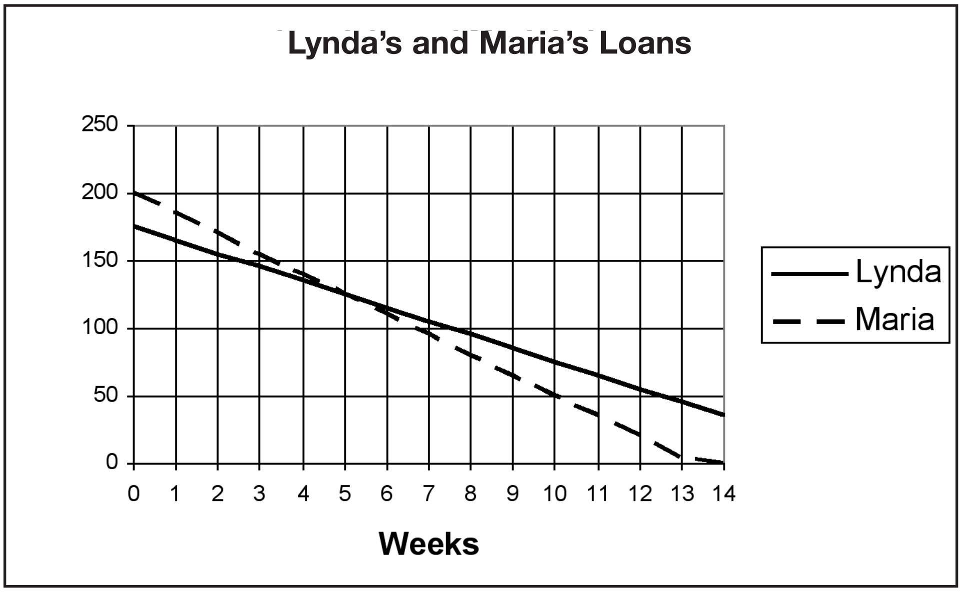
1. Which of the following lines has a greater slope? Explain. (4 points)



2. Determine the slope of each line using any method you wish. Verify your answer using any method. (6 points)

3. Use the following chart to answer the questions.

Lynda and Maria take out a loan from their parents. Maria borrowed $200 and pays back $15 a week. Lynda borrowed $175 and pays back $10 a week.



a. Are these lines increasing or decreasing? (5 points)
Does this make sense with the data? Explain.

b. At what week do they owe the same amount? What is the amount? (5 points)

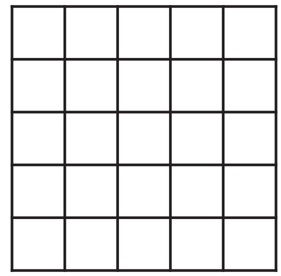
c. Who reaches zero first? (4 points)

d. How is your answer to c) above affected by the slope of the lines? (6 points)

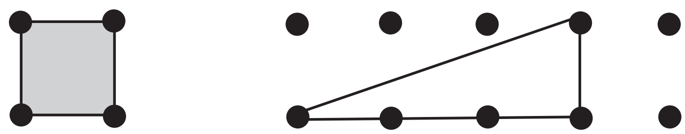
The STEM Project (1993-1998). *Book 1, Grade 6, Module 4.* MT: University of Montana.

Module E: (30 points)

1. Draw a figure on the centimeter graph paper that has an *area* of 6 sq units and a *perimeter* of 14 units. Use only squares, no diagonals. (4 points)



2. Find the *area* of the following triangle in *square units.* Remember that one square (4 dots) is the base unit. (4 points)



3. *Draw* each scenario and *calculate* the *perimeter.* Use your formulas from the handout where necessary. Drawings do not have to be to scale. Make sure to label. Use 3.14 for π where appropriate.

   a. Rectangle with a length of 3 inches and a width of 2 inches. (4 points)

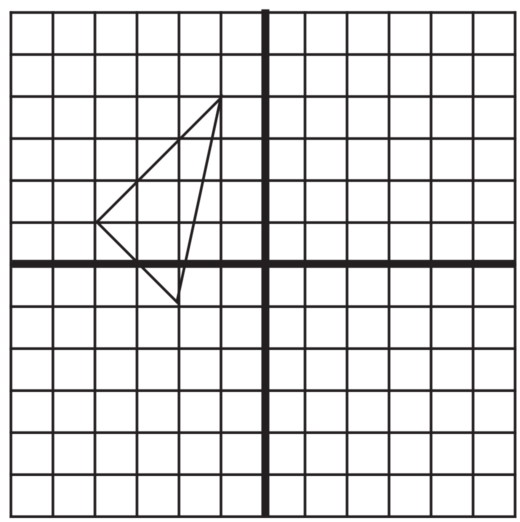
   b. Parallelogram with sides of 3 inches and 2 inches. (4 points)

4. *Draw* each scenario and *calculate* the *area.* Use your formulas from the handout where necessary. Drawings do not have to be to scale. Make sure to label. Use 3.14 for π where appropriate.

   a. Square with sides of 2 ft. (4 points)

   b. Circle with a diameter of 4 cm. (4 points)

5. *Reflect* the following figure across the *x axis.* Make sure to list the original coordinates and the new coordinates. (6 points)



Module E
Transparencies

Module E: Spatial Reasoning

The paraeducator will:

- Use concrete methods to determine the connections between perimeter and area
- Develop perimeter and area formulas for basic geometric shapes
- Explore perimeter and area concepts in relation to circles
- Use coordinate geometry to explain basic transformations

Perimeter and Area

Perimeter

Distance around a shape or region.

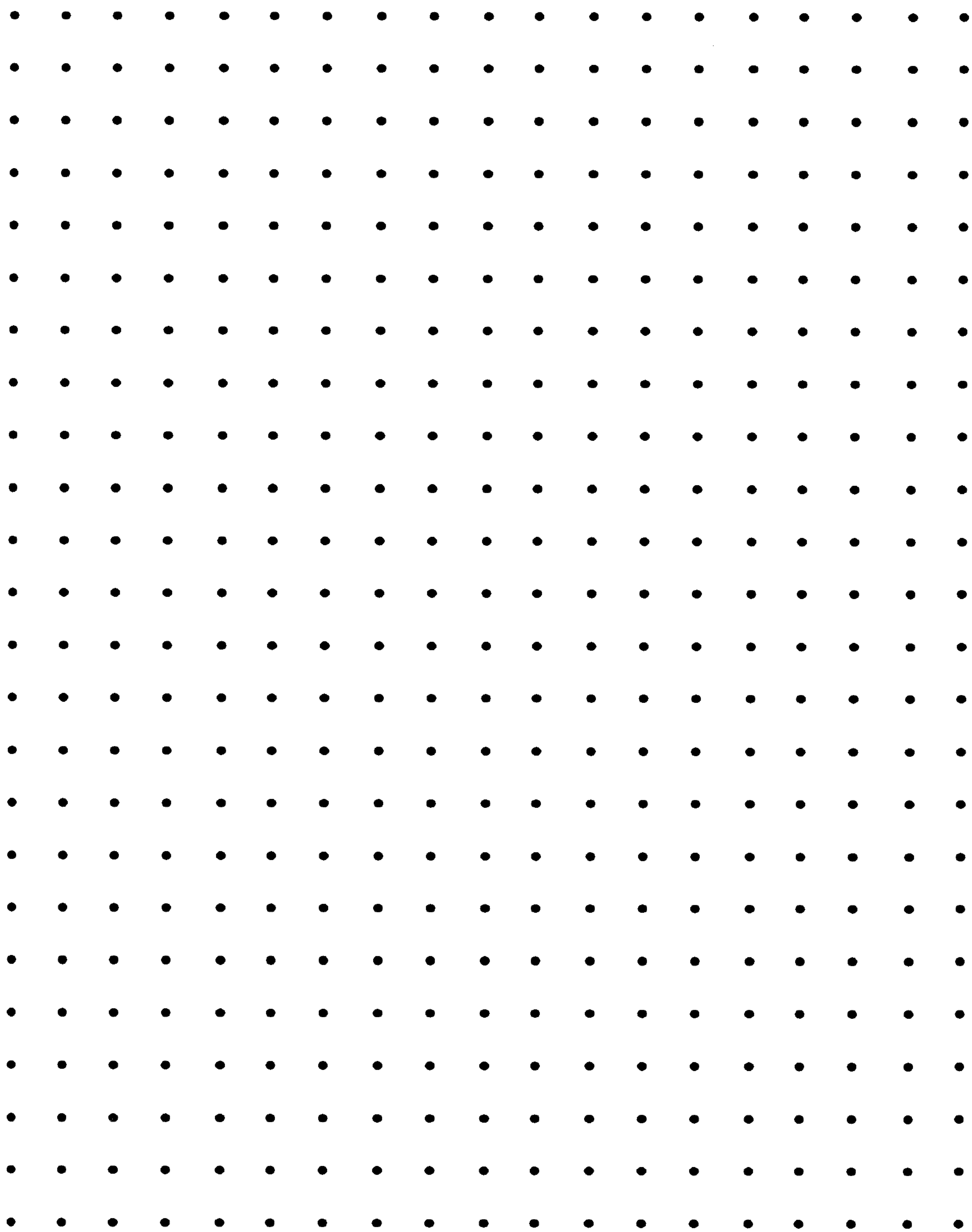
Examples: fencing, wallpaper border

Area

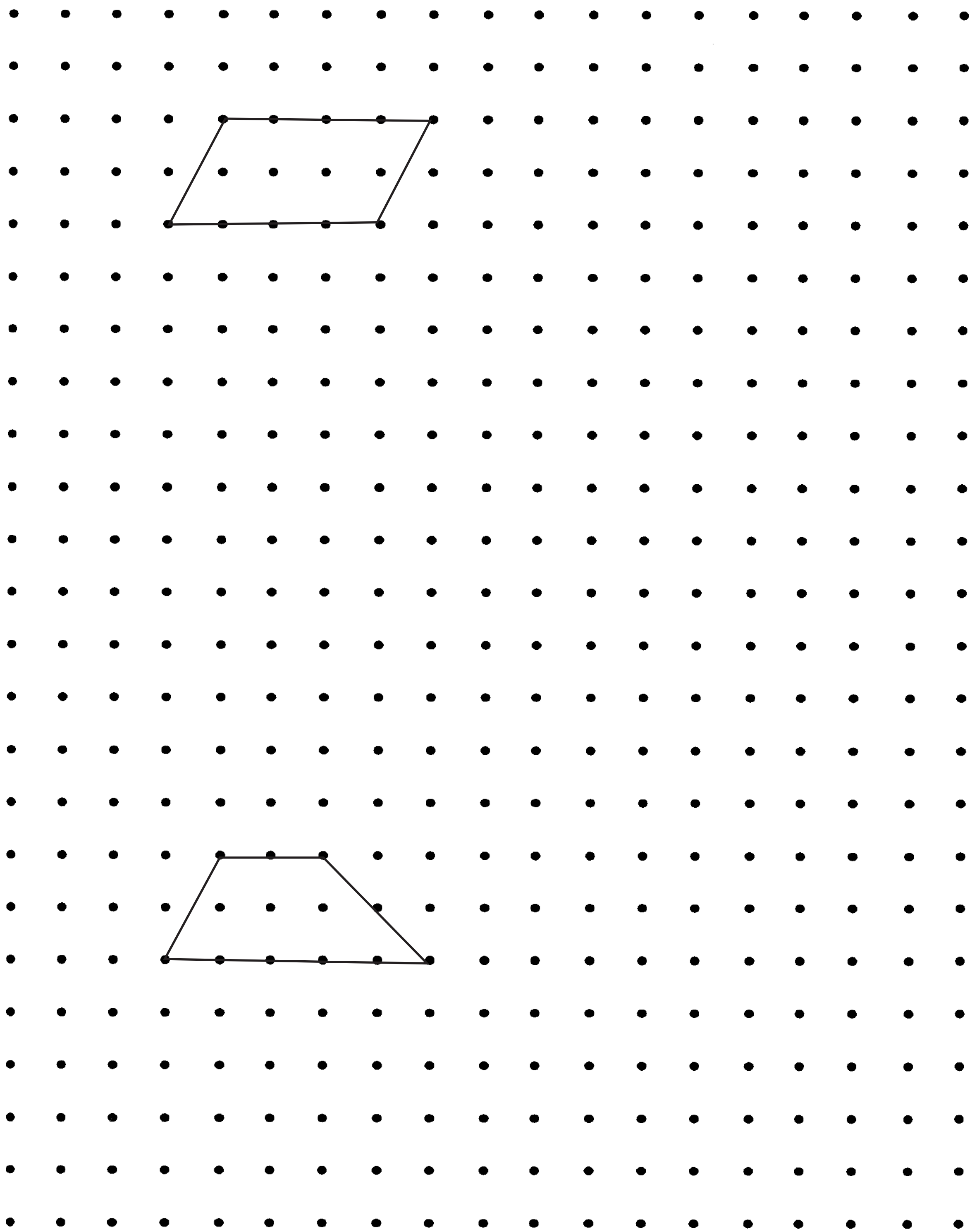
Amount of space inside a shape or region.

Examples: sod, paint, carpet

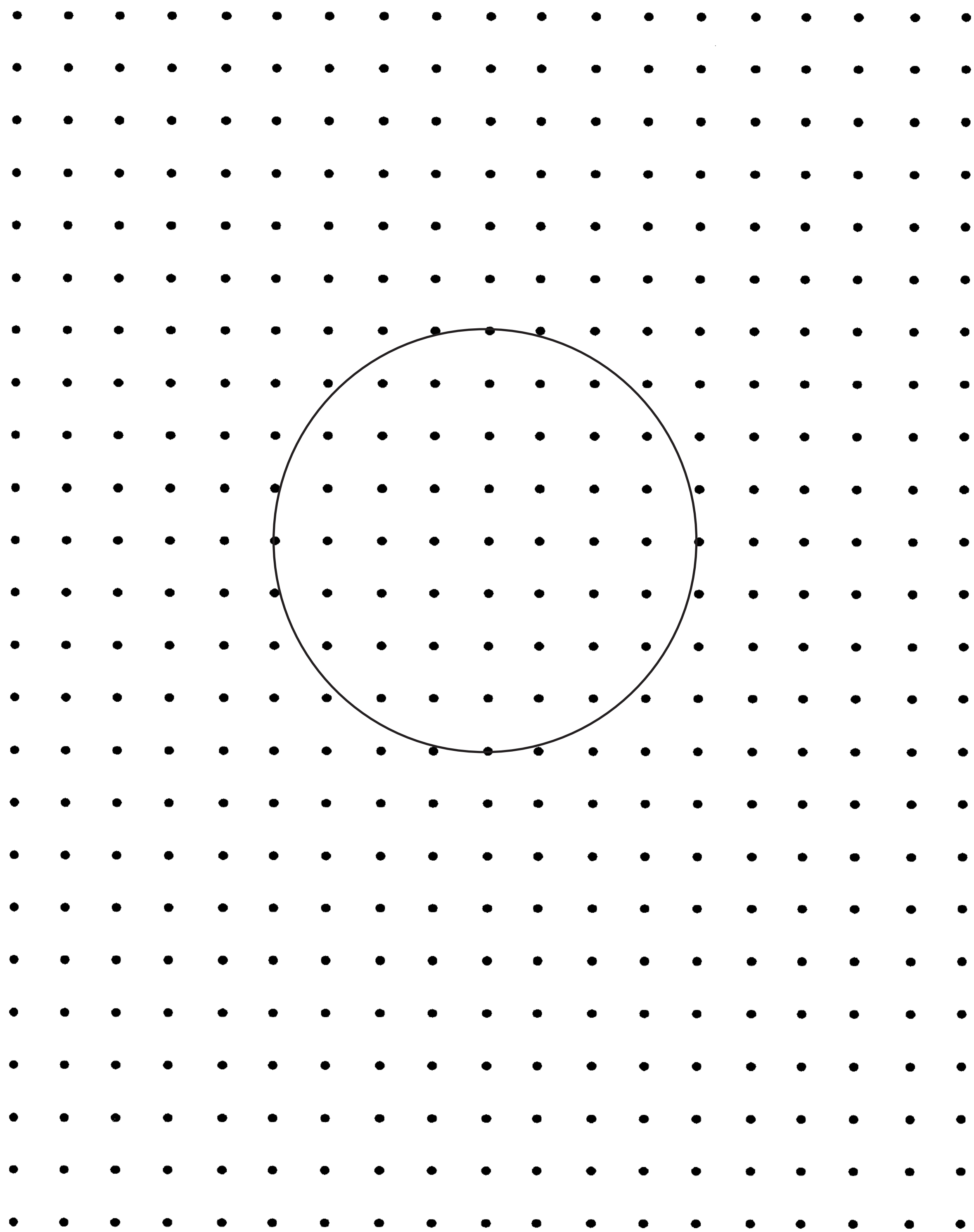
Dot Paper



Analyzing Area



Going in Circles



Transformations

Transformation:

A one-to-one correspondence between points in a plane.

Types:

Translation:

A motion of the figure that moves every point of the figure a specified distance in a specified direction along a straight line.

- Figures must be congruent
- Orientation does not change
- Position changes

Transformations

Reflection:

"Flips" an image about a line of reflection.

- Figures must be congruent
- Orientation and position changes

Size:

Reduces or increases a figure by a specified scale factor or ratio.

- Figures are *similar*
- Size changes by *scale factor* or ratio

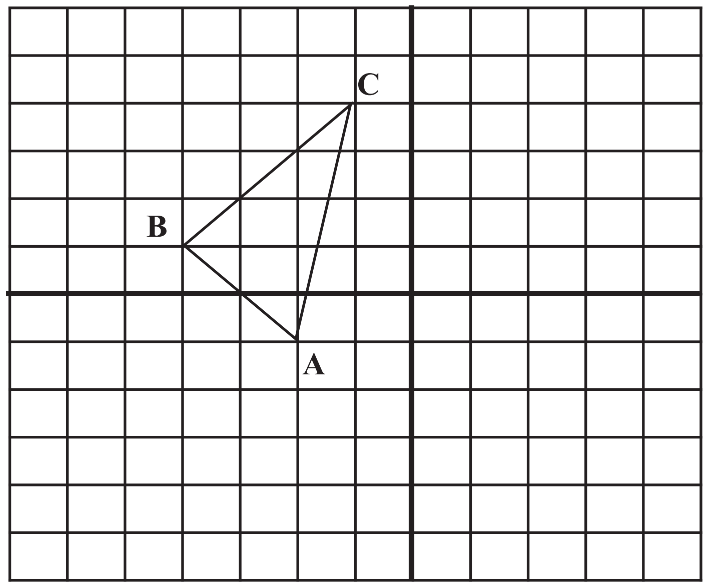
Scale Factor:

- A *scale factor* of two says that the resulting similar figure is twice as big as the original
- A *scale factor* of one half says that the resulting similar figure is half as big as the original

Move It

Complete each set of questions for the appropriate graph.

Set 1:



1. **Label the original coordinates for the associated points on the graph.**

2. **Translate the figure 5 units to the right. Draw the new figure.**

3. **Label the new points on the translated figure.**

4. **Which pattern do you see in the coordinates for a translation of five units to the right?**

5. **What pattern would you use with the coordinates to translate the original figure 3 units left? Give your new expected coordinates and test your pattern above by counting.**

    ***A*=** *A′*=
    ***B*=** *B′*=
    ***C*=** *C′*=

6. **What pattern would you use with the coordinates to translate the original figure 4 units down? Give your new expected coordinates and test your pattern above by counting.**

Move It

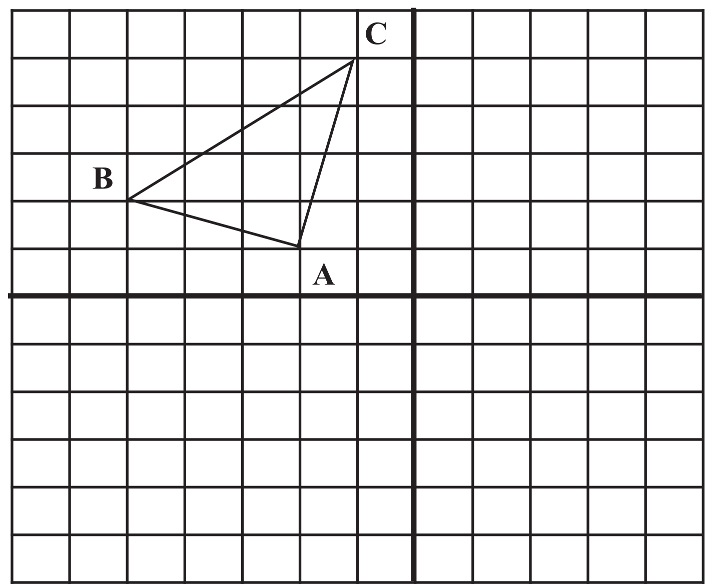
Set 2:

1. **Label the original coordinates for the associated points on the graph.**

2. **Reflect the figure over the y-axis. Draw the new figure.**

3. **Label the new points on the reflected figure.**

4. **What pattern do you see in the coordinates for a figure reflected over the y axis?**



5. **Which pattern would you use with the coordinates to reflect the original figure over the x axis? Give your new expected coordinates and test your pattern above by counting.**

A=	*A′*=
B=	*B′*=
C=	*C′*=

Move It

Set 3:

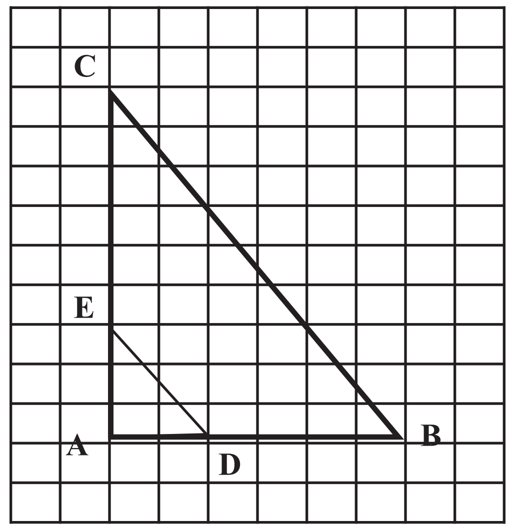
Triangles ABC and ADE are similar.

1. **Find ratio of** $\frac{AB}{AD}$**.**

2. **Find the ratio of** $\frac{AC}{AE}$ **.**

3. **Determine the scale factor from** *ADE* **to** *ABC***.**

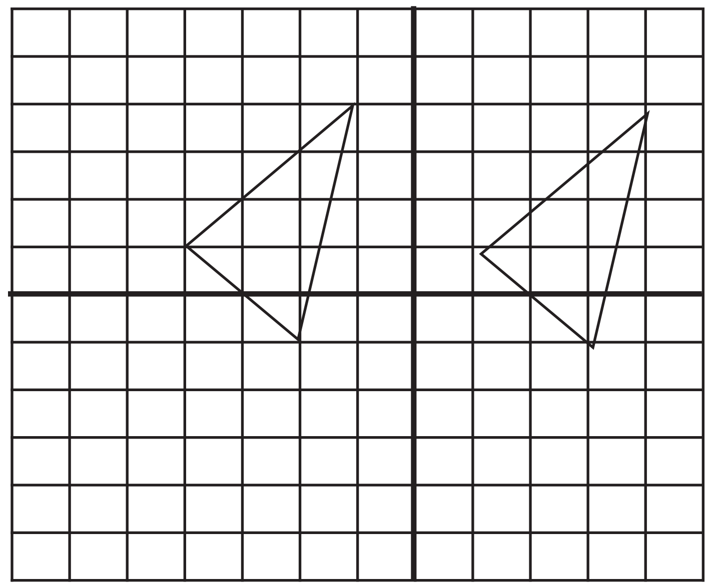
4. **Determine the scale factor from to** *ABC* **to** *ADE***.**



Move It – Answers

Complete each set of questions for the appropriate graph.

Set 1:



1. **Label the original coordinates for the associated points on the graph.**

   A (-2, -1), B (-4, 1), C (-1, 4)

2. **Translate the figure 5 units to the right. Draw the new figure.**

3. **Label the new points on the translated figure.**

   A′ **(3, -1),** ***B′*** **(1, 1),** ***C′*** **(4, 4)**

4. **Which pattern do you see in the coordinates for a translation of five units to the right?**

   5 was added to each x coordinate
   The y coordinates did not change for each point

5. **What pattern would you use with the coordinates to translate the original figure 3 units left? Give your new expected coordinates and test your pattern above by counting.**

   Subtract 3 from each x-coordinate
   A″ **(-5, -1),** ***B″*** **(-7, 1),** ***C″*** **(-4, 4)**

6. **What pattern would you use with the coordinates to translate the original figure 4 units down? Give your new expected coordinates and test your pattern above by counting.**

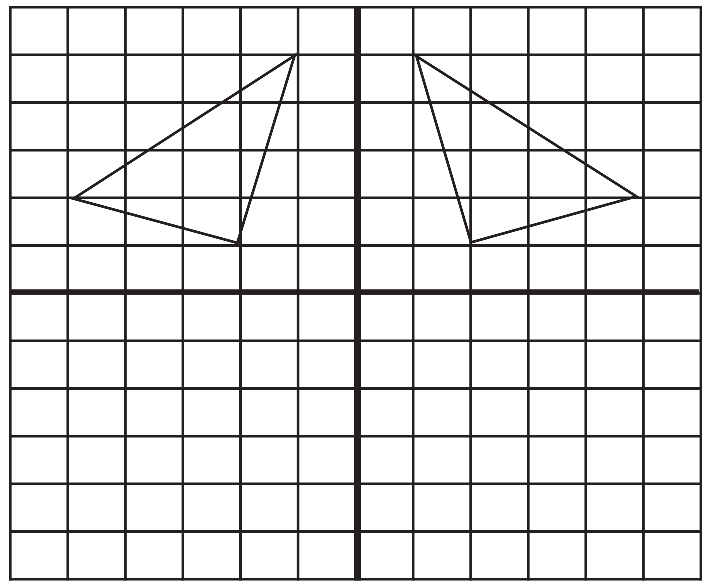
   Subtract 4 from each y coordinate
   A‴ **(-2, -5),** ***B‴*** **(-4, -3),** ***C‴*** **(-1, 0)**

Move It – Answers

Set 2:

1. **Label the original coordinates for the associated points on the graph.**

   A (-2, 1), B (-5, 2), C (-1, 5)



2. **Reflect the figure over the y-axis. Draw the new figure.**

3. **Label the new points on the reflected figure.**

   A′ **(2, 1),** *B′* **(5, 2),** *C′* **(1, 5)**

4. **What pattern do you see in the coordinates for a figure reflected over the y axis?**

   The sign of the x coordinate was the opposite.
   The y coordinates were the same for each point.

5. **Which pattern would you use with the coordinates to reflect the original figure over the x axis? Give your new expected coordinates and test your pattern above by counting.**

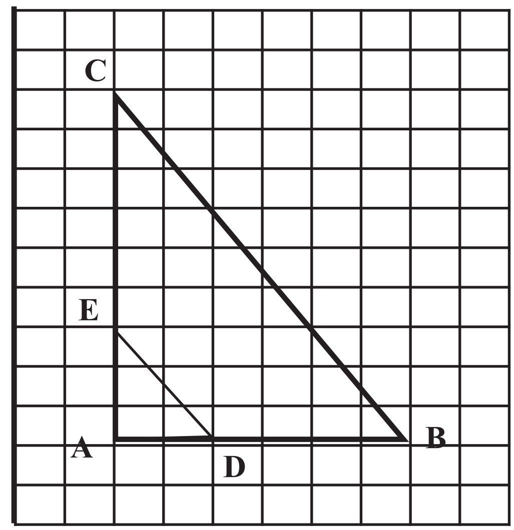
   Change the sign of the y coordinates to their opposite.
   Leave the x coordinate the same.

   A′ **(-2, -1),** *B′* **(-5, -2),** *C′* **(-1, -5)**

Move It – Answers

Set 3:

Triangles ABC and ADE are similar.



1. Find ratio of $\frac{AB}{AD}$.

$$\frac{AB}{AD} = \frac{6}{2} = 3$$

2. Find the ratio of $\frac{AC}{AE}$.

$$\frac{AC}{AE} = \frac{9}{3} = 3$$

3. Determine the scale factor from ADE **to** ABC.

Scale factor is 3

4. Determine the scale factor from to ABC **to** ADE.

Scale factor is 1/3

Algebraic Concepts and Spatial Reasoning

Grading Rubric and Grade Sheets

Grading Rubric for Algebraic Concepts and Spatial Reasoning

This rubric includes recommendations for grading:
1. Participation
2. Attendance
3. Assessment
4. Assignment
5. Final grade for academy

Grades are based upon a range of possible points earned:

Participation	Attendance	Assessment	Assignment	Total points possible
0-75	0-75	0-100	0-250	0-500

A	B	C	D	Failing
500-450	449-400	399-350	349-300	299 and below

Participation: Participants can earn up to **75** points for class participation. The instructor should consider the level of participation that occurs within smaller group settings as well as in larger group opportunities.

Attendance: Participants can earn up to **75** points for full attendance.

Assignment #1:
Probability and Integers (follows Module B). The assignment is worth a maximum of 130 points.

Assignment #2:
Algebra Skills (follows module C). This assignment is worth 120 points.

Final Assessment:
The final assessment (follows Module E) is an open-book test. It is worth 100 points.

Recording Sheets
Grading Rubric for Algebraic Concepts and Spatial Reasoning

Student	Participation	Attendance	Assessment	Assignment	Grand Total	Assigned Grade
1.						
2.						
3.						
4.						
5.						
6.						
7.						
8.						
9.						
10.						
11.						
12.						
13.						
14.						
15.						
16.						
17.						

Student	Participation	Attendance	Assessment	Assignment	Grand Total	Assigned Grade
18.						
19.						
20.						
21.						
22.						
23.						
24.						
25.						
26.						
27.						
28.						

References and Resources

Academy References

Billstein, R., Libeskind, S., & Lott. J. (2004). *A problem solving approach to mathematics for elementary school teachers* (8th ed.). Boston: Pearson, Addison Wesley.

Burns, M. (1992). *About teaching mathematics: K-8 resource.* White Plains, NY: Math Solutions Publications.

The STEM Project (1993-1998). *Book 1, Grade 6, Module 4*. Missoula: University of Montana.

National Council of Teachers of Mathematics. (2001). *Navigations series.* Reston, VA: Author.

Polya, G. (1957). *How to solve it.* Garden City, NY: Doubleday and Co.

Reys, R., Suydam, M., & Lindquist, M. (1992). *Helping children learn mathematics* (3rd ed.). Needham Heights, MA: Allyn and Bacon.

Product Resources

Pattern Blocks: individual or class sets may be purchased from:

EAI Education: www.eaieducation.com
ETA Cuisenaire. www.etacuisenaire.com

Web Resources for Practice and Further Lessons

A+ Math. www.aplusmath.com

This is an excellent site for premade flashcards, worksheets, and online games to help students practice basic skills from addition to basic prealgebra.

Purple Math. www.purplemath.com

This site offers clear and precise lessons starting with fraction/decimal concepts through high school algebra. Practice problems with immediate feedback are included.

National Council of Teachers of Mathematics. www.nctm.org

This site provides details on national math standards broken down by standard and grade-level bands. Free lesson plans and online activities to use with students are also available.

Supplies Needed for
ALGEBRAIC CONCEPTS

- One coin per each pair of students
- Pattern blocks on chips in 3 different colors (red, green, blue)
- Calculators
- Scissors
- Paper clips
- Blank transparencies
- Overhead markers in many colors
- Rulers or straight edges
- Yarn or string
- Circular objects to trace or geometry compasses
- Scrap paper with a blank side